

Liouville Type Models in Group Theory Framework

I. Finite-Dimensional Algebras

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ABSTRACT

In the series of papers we represent the “Whittaker” wave functional of $d + 1$ -dimensional Liouville model as a correlator in $d + 0$ -dimensional theory of the sine-Gordon type (for $d = 0$ and 1). Asymptotics of this wave function is characterized by the Harish-Chandra function, which is shown to be a product of simple Γ -function factors over all positive roots of the corresponding algebras (finite-dimensional for $d = 0$ and affine for $d = 1$). This is in nice correspondence with the recent results on 2- and 3-point correlators in $1 + 1$ Liouville model, where emergence of peculiar double-periodicity is observed. The Whittaker wave functions of $d + 1$ -dimensional non-affine (“conformal”) Toda type models are given by simple averages in the $d + 0$ dimensional theories of the affine Toda type. This phenomenon is in obvious parallel with representation of the free-field wave functional, which is originally a Gaussian integral over interior of a $d + 1$ -dimensional disk with given boundary conditions, as a (non-local) quadratic integral over the d -dimensional boundary itself. In the present paper we mostly concentrate on the finite-dimensional case. The results for finite-dimensional “Iwasawa” Whittaker functions were known, and we present their survey. We also construct new “Gauss” Whittaker functions.

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1 Introduction

Recent progress in the $2d$ Liouville theory [1, 2, 3] encourages one to make an attempt of full value in this long-standing problem of theoretical physics. It becomes more and more clear that a minor modification of the free field formalism is needed for the description of the Liouville correlators – a statement, long believed in within the framework of Hamiltonian reduction technique [4, 5, 6, 7]. The crucial point of this approach is the hidden group structure responsible for dynamical properties of the system.

Among other, this group structure provides a representation of the wave function of $d + 1$ -dimensional *integrable* theory in the form of d -dimensional functional integral (time excluded). Moreover, the asymptotics of the wave function (Harish-Chandra functions) can be further reduced to exponential $d-1$ -dimensional integrals (products). These asymptotics are of great importance, since their ratio determines the S -matrix of the theory. Alternatively, S -matrix is proportional to the 2-point function. As for the 3-point function (form-factor), the quantum mechanics is certainly not sufficient to fix it, because one needs to introduce additionally the particle creation vertex. But this is a specifics of the Liouville theory that the ratio of the two reflected 3-point functions is equal to the 2-point function [3]. This property allows one to restore the 3-point amplitude.

Therefore, our main purpose is to construct d -dimensional integral representation for the wave function and calculate its asymptotics. Technically the problem is a variation of the well-known free-field representation of Wess-Zumino-Novikov-Witten model and related coset models, which is in turn related to geometrical quantization on orbit manifolds (see [7] for details).

Although the problem of the main interest is the $d + 1$ -dimensional case, with $d \geq 1$ ($d = 1$ case being described by the affine algebras), we start our investigation in this paper with the simplest $d = 0$ case described by the finite-dimensional algebras. This case was much studied previously, and the paper can be partially taken as a survey. Our other purpose here is to prepare all the necessary background for the second paper in the series, devoted to the affine case.

The group theoretical approach to the Toda theories was considered by many authors starting from the first work of O. Bogoyavlensky [10]. It was clear from the very beginning that the open Toda models are related to the simple Lie algebras, while the periodic Toda models are coming from the affine algebras (see, for example, review [11]). Since an open Toda model can be derived by the Hamiltonian reduction from the free system on the cotangent bundle to a simple real Lie group [12, 13, 14], it is natural to suggest that the quantum theory of the model is based on the irreducible unitary representations of this group. It turned out that the relevant representations belong to the so-called Whittaker model [15, 16, 17, 18]. The idea of applying these representations to the quantization of the open Toda is due to Kostant (unpublished). It was elaborated in detail later by Semenov-Tian-Shansky [19]. This approach allowed one to give explicit expressions for the wave functions and to find the S -matrix. In the group theoretical terms the wave functions of the open Toda model are related in a very simple way to the Whittaker function, while the S matrix is defined by the Harish-Chandra function. This function determines the Plancherel measure on the set of irreducible unitary representations contributing to the Whittaker model.

There are some interesting results for this class of models, which were obtained

beyond the group-theoretical approach [22, 23, 24, 25], but they dealt with the two- and three-body problems only.

Of the main interest, however, is the Liouville field theory, which on the classical level is the reduced system, coming from the cotangent bundle to the central extended loop group $L(\widehat{SL}_2)$ [26]. Therefore, it should be suggested that some analog of the Whittaker model for $L(\widehat{SL}_2)$ is responsible for the quantization of the Liouville field theory. More concretely, we present in the second paper of the series the group-theoretical derivation of formulas for correlators obtained recently in [2, 3]. Our approach is very close to the philosophy of "geometrization" of the scattering in quantum integrable systems advocated by Freund and Zabrodin [27]. They suggested that there should be a correspondence between S -matrices and Harish-Chandra functions for some groups or their suitable deformations (quantum, p -adic, elliptic etc.).

As we mentioned above there is another theory – the quantum periodic Toda lattice – which is also related to the representations of the affine algebras. From this point of view, it was considered in the last paper of [28]. Some results for the two and three body quantum periodic Toda chains [22] have a similar interpretation. However, we are not aware of any counterparts of the Whittaker models for the affine groups.

The Whittaker model was applied to construct automorphic representations for the groups over the ring of adels [29, 30]. Later it was used in [31] in the construction of geometric Langlands correspondence in the case $GL(2)$.

The Whittaker function for the quantum Lorentz group was defined in [32]. It arises there as a wave function in some integrable discretization of the Liouville quantum mechanics.

Considerable part of the material we consider in detail in this paper, i.e.

related to the finite dimensional case is well known and can be found in [19] and [18]. Our presentation is customized to the extension to the affine case. We add also some new results (Gauss-Whittaker function, modification of the Harish-Chandra function), which will be used in the infinite-dimensional situation. Furthermore, due to a parallel between the quantum open Toda model and quantum Calogero-Sutherland model, which follows from their group theoretical origin, we briefly describe the later model and the corresponding spherical model of irreducible representations. More concretely, the wave functions of the quantum Calogero-Sutherland models are gauge transform of the zonal spherical functions on a real simple group G . The later are matrix elements in the unitary irreducible representations with both the bra and the ket vectors being K -invariant, where K is a maximal compact subgroup of G . This definition is based on the Cartan decomposition of G . The wave functions for the Toda models are gauge transform of the Whittaker functions. They are defined as matrix elements in the same representations with pairing between K -invariant and N -covariant states, (N is the positive nilpotent subgroup) and are related to the Iwasawa decomposition. We also consider the pairing between N -covariant and \bar{N} -covariant states, (\bar{N} is the negative nilpotent subgroup), which is coming from the Gauss decomposition. Thereby, we cover all possible non-trivial reductions to the Cartan subgroups.

The paper organized as follows. First, we discuss the general construction and point out some important relations with the affine case. Second, in Part 1 we consider the particular examples of $SL(2)$, $SL(3)$ and $SL(N)$ in detail. At last, the second Part is devoted to the general construction.

Throughout the paper we use standard notations and constructions of the group theory without additional references. One can use, say, [20, 21] as stan-

dard text-books.

2 General scheme and comments

Let us explain the idea of group theory construction of the wave functions in the very general terms.

1) Let $g(\xi|T) \in \mathcal{A}_G(\xi) \otimes \mathcal{U}_G$ be the “universal group element” of a Lie group G (perhaps, quantum group) [33, 34], where ξ parametrizes somehow the group manifold, and T are generators of G in some (not obligatory irreducible) representation, their only property being $[T^a, T^b] = f^{abc}T^c$.

2) For every given parametrization $\{\xi\}$, one can introduce two sets of differential (difference) operators $\mathcal{D}_{R,L}(\xi)$, such that

$$\begin{aligned}\mathcal{D}_L^a(\xi)g(\xi|T) &= T^a g(\xi|T), \\ \mathcal{D}_R^a(\xi)g(\xi|T) &= g(\xi|T)T^a.\end{aligned}\tag{2.1}$$

These operators satisfy the obvious commutation relations:

$$\begin{aligned}[\mathcal{D}_L^a, \mathcal{D}_L^b] &= -f^{abc}\mathcal{D}_L^c, \\ [\mathcal{D}_R^a, \mathcal{D}_R^b] &= f^{abc}\mathcal{D}_R^c, \\ [\mathcal{D}_L^a, \mathcal{D}_R^b] &= 0.\end{aligned}\tag{2.2}$$

3) For given representation \mathcal{R} , scalar product $\langle \cdot | \cdot \rangle$ and two elements $\langle \psi_L |$ and $|\psi_R \rangle$, one can construct matrix element

$$F_{\mathcal{R}}(\xi|\psi_L, \psi_R) = \langle \psi_L | g(\xi|T) | \psi_R \rangle.\tag{2.3}$$

Then, action of any combination of differential operators \mathcal{D}_R on F inserts the same combination of generators T to the right of $g(\xi|T)$. If $|\psi_R \rangle$ happens to be

an eigenvector of this combination of generators, the corresponding F provides a solution to the differential equation.

4) Of special interest are Casimir operators, since $|\psi_R\rangle$, which are their eigenvectors can be easily described as elements of irreducible representations, or linear combinations of such elements in degenerate cases when different representations have the same values of Casimir operators.

5) Quadratic Casimir operators, when expressed through \mathcal{D}_R , can serve as Laplace operators, leading to some important Schrödinger equations, of which that for the Liouville theory is a typical example. In order to obtain the Liouville model, one should impose additional constraints on the states $\langle\psi_L|$ and $|\psi_R\rangle$, what corresponds to the Hamiltonian reduction of the free motion on the group manifold of G . Such a reduction is usually associated with some decomposition of G into a product of subgroups (like Gauss or Iwasawa decompositions), moreover, different decompositions can give rise to equivalent reductions. Essentially, reduction allows one to eliminate the dependence of $F(\xi)$ on some of the coordinates ξ . In the case of finite-dimensional Lie groups, the remaining coordinates can be made just those on the Cartan torus, while for affine algebras it is more interesting to keep the dependence of all the diagonal matrices.

Let us stress that there are, at least, three different ways to obtain the reduced Hamiltonians. The first one described above is to express the Casimir operator through \mathcal{D}_R and impose the reduction condition after this. However, one can calculate the reduced Hamiltonian immediately from the matrix element $F_R(\xi|\psi_L, \psi_R)$ inserting the Casimir operator into $\langle\psi_L|g(\xi|T)|\psi_R\rangle$. We illustrate these procedures with concrete examples in Part I. There is also another way, which we use in Part II, based on the observation that the second

Casimir operator coincides with the Laplace-Beltrami operator, which can be calculated making use of the Killing metric in the concrete (Iwasawa or Gauss) coordinates. This way is however inconvenient for constructing higher Casimir operators.

Now we briefly comment on the general procedure postponing the detailed discussion till Part II, while illustrative examples are considered in Part I.

First of all, let us remark that the differential operators \mathcal{D}_L and \mathcal{D}_R realize respectively the left and the right regular representations of the algebra of G (in fact, the left one is anti-representation), that can be also given by the action on the space A_G of functions on the group:

$$\pi_{reg}^L(h)f(g) = f(hg), \quad \pi_{reg}^R(h)f(g) = f(gh), \quad g, h \in G. \quad (2.4)$$

Manifestly these operators can be constructed in the following way. Let us consider the universal group element g (it is sufficient to consider g only in the fundamental representation, since matrix elements in the fundamental representation are generating elements of the whole algebra A_G , and the action of the group can be extended to the whole algebra A_G making use of comultiplication) and the (formal) differential operator d acting as the full derivative on functions of ξ_i , i.e. $d \equiv \sum_i d\xi_i \frac{\partial}{\partial \xi_i}$. Then, one may calculate, say, $g^{-1} \cdot dg$ (Maurer-Cartan form) and expand it in the generators of the algebra (since the fundamental representation is sufficient to fix the coefficients $c_{a,i}$, the calculations are very simple):

$$g^{-1} \cdot dg = \sum_{a,i} c_{a,i} T_a d\xi_i, \quad (2.5)$$

i.e.

$$dg = \sum_{a,i} c_{a,i} (\mathcal{D}_R^a g) d\xi_i. \quad (2.6)$$

Now one reads off the manifest form of the differential operators \mathcal{D}_R from this expression. Analogously, one can calculate \mathcal{D}_L .

Our second remark concerns different possible reductions. In fact, there are three principally different kinds of reductions induced by the Iwasawa and Gauss decompositions and by reducing to the radial part of the Cartan decomposition. The first two reductions lead to the same Liouville wave functions while the last one – to the zonal spherical functions. In this paper we are interested both in the Gauss decomposition and in the Iwasawa one. One of the main points of our interest are the asymptotics of the Liouville wave function – Harish-Chandra functions. The number of different asymptotics is equal to the number of elements of the Weyl group. The ratios of them give the S -matrices and, equivalently, 2-point functions. We demonstrate in this paper that the Harish-Chandra functions for the finite-dimensional groups are equal to

$$c(\boldsymbol{\lambda}) = \prod_{\boldsymbol{\alpha} \in \Delta^+} \frac{1}{\Gamma(1 + \boldsymbol{\lambda} \cdot \boldsymbol{\alpha})} \quad (2.7)$$

and all the other are obtained by the action of the Weyl group on $\boldsymbol{\lambda}$.

This form can be easily continued to the affine case. Indeed, let us consider $\widehat{SL(2)}$ case, for the sake of simplicity. Then, the system of positive roots is

$$\boldsymbol{\alpha}_0 + n(\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1), \quad n(\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1), \quad \boldsymbol{\alpha}_1 + n(\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1), \quad n = 0, 1, 2, \dots \quad (2.8)$$

Let

$$\boldsymbol{\lambda} \cdot \boldsymbol{\alpha}_0 = \frac{1}{2} - p + \tau, \quad \boldsymbol{\lambda} \cdot \boldsymbol{\alpha}_1 = -\frac{1}{2} + p. \quad (2.9)$$

Then, one should shift the argument of the Γ -functions to $1/2$ because of the

affine situation to obtain

$$c(\boldsymbol{\lambda}) = \prod_{n \geq 0} \Gamma^{-1}(p + n\tau) \prod_{n \geq 1} \Gamma^{-1}(n\tau) \Gamma^{-1}(1 - p + n\tau). \quad (2.10)$$

This expression still requires a careful regularization, but all the infinite products cancel from the corresponding reflection S -matrix (2-point function)¹

$$S(p) = \frac{c(-p)}{c(p)} = \frac{\Gamma(1+p)\Gamma(1+\frac{p}{\tau})}{\Gamma(1-p)\Gamma(1-\frac{p}{\tau})}. \quad (2.11)$$

This expression is to be compared with the formulas for the 2-point functions obtained in papers [2, 3] in a very different way². In the second paper of the series we are going to discuss the properties of the functions of such a type.

Most important, (2.10) exhibits the double-(quasi)periodicity property – it can be symbolically (modulo requested regularizations) represented as

$$c(\boldsymbol{\lambda}) \sim \prod_{m,n} (p + m + n\tau). \quad (2.12)$$

From this observation it is just a step to consider the most fundamental object

$$i(\boldsymbol{\lambda}) = i(p, \tau) \sim \prod_{\substack{\text{one} \\ \text{quadrant}}} (p + m + n\tau). \quad (2.13)$$

Indeed, $i(p)$ is a building block for both the elliptic theta-functions,

$$\theta(p + \frac{1}{2} + \frac{\tau}{2}, \tau) \sim i(p, \tau) i(-p, \tau) i(p, -\tau) i(-p, -\tau) \quad (2.14)$$

and the quantum exponentials,

¹Throughout the paper we omit from the S -matrix a trivial factor depending on the cosmological constant μ .

²In notations of [3], $p = 2iP/b$ and $\tau = b^2$.

$$e_q(e^{2\pi ip}) \sim \frac{1}{i(p, \tau)i(-p, -\tau)}, \quad 1 = e^{i\pi\tau}. \quad (2.15)$$

Part I

Particular groups

1 $SL(2)$

1.1 Notations

The Lie algebra is defined by the relations:

$$[T_+, T_-] = T_0, \quad [T_\pm, T_0] = \mp 2T_\pm. \quad (1.1)$$

The fundamental representation:

$$T_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (1.2)$$

The quadratic Casimir operator:

$$C = (T_- T_+ + T_+ T_-) + \frac{1}{2} T_0^2 = 2T_- T_+ + T_0 + \frac{1}{2} T_0^2. \quad (1.3)$$

Different parametrizations of the group element:

$$\begin{aligned} g_I(\theta, \phi, \chi|T) &= e^{\theta T_2} e^{\phi_I T_0} e^{\chi_I T_+}, \\ g_G(\psi, \phi, \chi|T) &= e^{\psi T_-} e^{\phi T_0} e^{\chi T_+}. \end{aligned} \quad (1.4)$$

Group element in the fundamental representation:

$$\begin{aligned} g_I &= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} e^{\phi_I} & 0 \\ 0 & e^{-\phi_I} \end{pmatrix} \begin{pmatrix} 1 & \chi_I \\ 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} e^{\phi_I} \cos \theta & \chi_I e^{\phi_I} \cos \theta + e^{-\phi_I} \sin \theta \\ -e^{\phi_I} \sin \theta & -\chi_I e^{\phi_I} \sin \theta + e^{-\phi_I} \cos \theta \end{pmatrix}, \end{aligned} \quad (1.5)$$

$$g_G = \begin{pmatrix} 1 & 0 \\ \psi & 1 \end{pmatrix} \begin{pmatrix} e^\phi & 0 \\ 0 & e^{-\phi} \end{pmatrix} \begin{pmatrix} 1 & \chi \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^\phi & \chi e^\phi \\ \psi e^\phi & \psi \chi e^\phi + e^{-\phi} \end{pmatrix}. \quad (1.6)$$

Connection between different parametrizations:

$$\begin{aligned} \psi &= -\tan \theta, \\ e^\phi &= e^{\phi_I} \cos \theta, \\ \chi &= \chi_I + e^{-2\phi_I} \tan \theta. \end{aligned} \quad (1.7)$$

1.2 Representations

1.2.1 Regular representations

Right and left regular representations can be read off using the manifest expressions for the "currents" $g^{-1} \cdot dg$ and $dg \cdot g^{-1}$. In the Iwasawa case, these are

$$\begin{aligned} g_I^{-1} dg_I &= \begin{pmatrix} d\phi_I + e^{2\phi_I} \chi_I d\theta & d\chi_I + 2\chi_I d\phi_I + (\chi_I^2 e^{2\phi_I} + e^{-2\phi_I}) d\theta \\ -e^{2\phi_I} d\theta & -d\phi_I - e^{2\phi_I} \chi_I d\theta \end{pmatrix}, \\ dg_I g_I^{-1} &= \begin{pmatrix} \cos 2\theta d\phi_I + \frac{1}{2} e^{2\phi_I} \sin 2\theta d\chi_I & e^{2\phi_I} \cos^2 \theta d\chi_I - \sin 2\theta d\phi_I + d\theta \\ -e^{2\phi_I} \sin^2 \theta d\chi_I - \sin 2\theta d\phi_I - d\theta & -\cos 2\theta d\phi_I - \frac{1}{2} e^{2\phi_I} \sin 2\theta d\chi_I \end{pmatrix}, \end{aligned} \quad (1.8)$$

while, for the Gauss case, one obtains

$$\begin{aligned} g^{-1} dg &= \begin{pmatrix} -e^{2\phi} \chi d\psi + d\phi & -e^{2\phi} \chi^2 d\psi + 2\chi d\phi + d\chi \\ e^{2\phi} d\psi & e^{2\phi} \chi d\psi - d\phi \end{pmatrix}, \\ dg \cdot g^{-1} &= \begin{pmatrix} -e^{2\phi} \psi d\chi + d\phi & e^{2\phi} d\chi \\ -e^{2\phi} \psi^2 d\chi + 2\psi d\phi + d\psi & e^{2\phi} \psi d\chi - d\phi \end{pmatrix}. \end{aligned} \quad (1.9)$$

Using these formulas, one can easily read off the following expressions

$$\begin{aligned} \frac{\partial g_I}{\partial \phi_I} &= g(T_0 + 2\chi_I T_+) = (\cos 2\theta T_0 - \sin 2\theta T_+ - \sin 2\theta T_-)g, \\ \frac{\partial g_I}{\partial \theta} &= g(e^{2\phi_I} \chi_I T_0 + (\chi_I^2 e^{2\phi_I} + e^{-2\phi_I}) T_+ - e^{2\phi_I} T_-) = (T_+ - T_-)g, \\ \frac{\partial g_I}{\partial \chi_I} &= gT_+ = \left(\frac{1}{2} e^{2\phi_I} \sin 2\theta T_0 + e^{2\phi_I} \cos^2 \theta T_+ - e^{2\phi_I} \sin^2 \theta T_- \right) g \end{aligned} \quad (1.10)$$

and

$$\begin{aligned} \frac{\partial g}{\partial \phi} &= g(T^0 + 2\chi T^+) = (T^0 + 2\psi T^-)g, \\ \frac{\partial g}{\partial \chi} &= gT^+ = e^{2\phi} (T^+ - \psi T^0 - \psi^2 T^-)g, \\ \frac{\partial g}{\partial \psi} &= g(-\chi^2 T^+ - \chi T^0 + T^-)e^{2\phi} = T^-g. \end{aligned} \quad (1.11)$$

Right regular representation

In the Iwasawa case, one gets using (1.10)

$$\begin{aligned} \mathcal{D}_R^+ &= \frac{\partial}{\partial \chi_I}, \quad \mathcal{D}_R^0 = \frac{\partial}{\partial \phi_I} - 2\chi_I \frac{\partial}{\partial \chi_I}, \\ \mathcal{D}_R^- &= (e^{-4\phi_I} - \chi_I^2) \frac{\partial}{\partial \chi_I} + \chi_I \frac{\partial}{\partial \phi_I} - e^{-2\phi_I} \frac{\partial}{\partial \theta}. \end{aligned} \quad (1.12)$$

In the Gauss case, one should use (1.11)

$$\mathcal{D}_R^+ = \frac{\partial}{\partial \chi}, \quad \mathcal{D}_R^0 = -2\chi \frac{\partial}{\partial \chi} + \frac{\partial}{\partial \phi}, \quad \mathcal{D}_R^- = e^{-2\phi} \frac{\partial}{\partial \psi} + \chi \frac{\partial}{\partial \phi} - \chi^2 \frac{\partial}{\partial \chi}. \quad (1.13)$$

Left regular representation

Analogously to the previous paragraph, one gets in the Iwasawa case

$$\begin{aligned} \mathcal{D}_L^- &= \cos 2\theta e^{-2\phi_I} \frac{\partial}{\partial \chi} - \frac{1}{2} \sin 2\theta \frac{\partial}{\partial \phi_I} - \cos^2 \theta \frac{\partial}{\partial \theta}, \\ \mathcal{D}_L^0 &= 2 \sin 2\theta e^{-2\phi_I} \frac{\partial}{\partial \chi} + \cos 2\theta \frac{\partial}{\partial \phi_I} - \sin 2\theta \frac{\partial}{\partial \theta}, \\ \mathcal{D}_L^+ &= \cos 2\theta e^{-2\phi_I} \frac{\partial}{\partial \chi} - \frac{1}{2} \sin 2\theta \frac{\partial}{\partial \phi_I} + \sin^2 \theta \frac{\partial}{\partial \theta}. \end{aligned} \quad (1.14)$$

In the Gauss case

$$\mathcal{D}_L^- = \frac{\partial}{\partial \psi}, \quad \mathcal{D}_L^0 = -2\psi \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \phi}, \quad \mathcal{D}_L^+ = -\psi^2 \frac{\partial}{\partial \psi} + \psi \frac{\partial}{\partial \phi} + e^{-2\phi} \frac{\partial}{\partial \chi}. \quad (1.15)$$

1.2.2 Highest weight representation

We consider the principal (spherical) series of representations, induced by the one-dimensional representations of the Borel subgroup. The space of representation is functions of one real variable x and matrix elements are defined by integrals with the flat measure. The action of the group is given by differential operators:

$$T_+ = \frac{\partial}{\partial x}, \quad T_0 = -2x \frac{\partial}{\partial x} + 2j, \quad T_- = -x^2 \frac{\partial}{\partial x} + 2jx. \quad (1.16)$$

In this paper we consider only unitary highest weight irreducible representations. The requirement of unitarity is crucial for our needs since it makes all integrals convergent. For the series of the representations that we consider in this paper, unitarity implies that $j + \frac{1}{2}$ is pure imaginary. Analogously, in the case of general $SL(N)$, only pure imaginary values of $j_i + \frac{1}{2}$ (i.e. $\mathbf{j} + \boldsymbol{\rho}$) are admissible.

1.3 Hamiltonian

1.3.1 Reducing from the regular representation

Calculating either in the left regular representation, or in the right one, we obtain for the quadratic Casimir operator in the Iwasawa case

$$\begin{aligned} C &= \frac{1}{2} \frac{\partial^2}{\partial \phi_I^2} + \frac{\partial}{\partial \phi_I} + 2e^{-4\phi_I} \frac{\partial^2}{\partial \chi_I^2} - 2e^{-2\phi_I} \frac{\partial^2}{\partial \chi_I \partial \theta} = \\ &= e^{-\phi} \left(\frac{1}{2} \frac{\partial^2}{\partial \phi_I^2} + 2e^{-4\phi_I} \frac{\partial^2}{\partial \chi_I^2} - 2e^{-2\phi_I} \frac{\partial^2}{\partial \chi_I \partial \theta} - \frac{1}{2} \right) e^{\phi}. \end{aligned} \quad (1.17)$$

Its eigenvalue in the spin- j representation is $2(j^2 + j)$. Therefore, the matrix element $F_I = \langle \psi_L | g(\theta, \phi_I, \chi_I) | \psi_R \rangle$ with $\langle \psi_L |$ and $|\psi_R \rangle$ belonging to the spin- j representation satisfies the equation

$$CF_I = 2(j^2 + j)F_I. \quad (1.18)$$

We choose matrix element F_I to lie in the (highest weight) spin- j (irreducible) representation of the principal (spherical) series of $SL(2, \mathbf{R})$. The Liouville Hamiltonian is obtained from this Casimir operator after the reduction is imposed

$$\frac{\partial}{\partial \chi_I} F_I = i\mu F_I, \quad \frac{\partial}{\partial \theta} F_I = 0, \quad (1.19)$$

which implies (see (1.12) and (1.14))

$$T_+ |\psi_R \rangle = i\mu |\psi_R \rangle, \quad \langle \psi_L | T_2 = 0 \quad (\text{i.e. } \langle \psi_L | T_- = \langle \psi_L | T_+). \quad (1.20)$$

Then, the Hamiltonian is equal to

$$H = \frac{1}{2} \frac{\partial^2}{\partial \phi_I^2} + \frac{\partial}{\partial \phi_I} - 2\mu^2 e^{-4\phi_I} \quad (1.21)$$

and the function $\Psi_I(\phi_I) = e^{\phi_I} F_I$ satisfies the following Schrödinger equation

$$\left[\frac{1}{2} \frac{\partial^2}{\partial \phi_I^2} - 2\mu^2 e^{-4\phi_I} \right] \Psi_I(\phi_I) = 2 \left(j + \frac{1}{2} \right)^2 \Psi_I(\phi_I). \quad (1.22)$$

In the Gauss case the quadratic Casimir operator is

$$C = \frac{1}{2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial}{\partial \phi} + 2e^{-2\phi} \frac{\partial^2}{\partial \psi \partial \chi}. \quad (1.23)$$

Then, the reduction conditions are

$$\frac{\partial}{\partial \chi} F_G = i\mu_R F_G, \quad \frac{\partial}{\partial \psi} F_G = i\mu_L F_G, \quad (1.24)$$

i.e. (see (1.13) and (1.15))

$$T_+|\psi_R\rangle = i\mu_R|\psi_R\rangle, \quad \langle\psi_L|T_- = i\mu_L\langle\psi_L|. \quad (1.25)$$

The Hamiltonian is equal to

$$H = \frac{1}{2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial}{\partial \phi} - 2\mu_R\mu_L e^{-2\phi} \quad (1.26)$$

and the function $\Psi_G(\phi) = e^\phi F_G$ satisfies the following Schrödinger equation

$$\left[\frac{1}{2} \frac{\partial^2}{\partial \phi^2} - 2\mu_R\mu_L e^{-2\phi} \right] \Psi_G(\phi) = 2 \left(j + \frac{1}{2} \right)^2 \Psi_G(\phi), \quad (1.27)$$

Comment. Two Schrödinger equations (1.22) and (1.27) are related by the replace $j + \frac{1}{2} \rightarrow 2j + 1$ and the appropriate rescalings of μ 's and ϕ 's.

1.3.2 Immediate matrix element reduction

Using the representation (1.5) and conditions (1.20), we can immediately obtain Liouville equation with Hamiltonian (1.21) in the Iwasawa case³

$$\begin{aligned} 2j(j+1)F_I^{(j)} &\equiv 2j(j+1) \langle\psi_L|e^{\phi_I T_0}|\psi_R\rangle_j = \langle\psi_L|e^{\theta T_-}e^{\phi_I T_0}e^{\chi_I T_+}\hat{C}|\psi_R\rangle_j = \\ &= \langle\psi_L|e^{\phi_I T_0}(2T_-T_+ + T_0 + \frac{1}{2}T_0^2)|\psi_R\rangle_j = \\ &= 2i\mu e^{-2\phi_I} \langle\psi_L|T_-e^{\phi_I T_0}T_+|\psi_R\rangle_j + \left(\frac{\partial}{\partial \phi_I} + \frac{1}{2} \frac{\partial^2}{\partial \phi_I^2} \right) \langle\psi_L|e^{\phi_I T_0}|\psi_R\rangle_j = \\ &= \left(-2\mu^2 e^{-4\phi_I} + \frac{\partial}{\partial \phi_I} + \frac{1}{2} \frac{\partial^2}{\partial \phi_I^2} \right) \langle\psi_L|e^{\phi_I T_0}|\psi_R\rangle_j. \end{aligned} \quad (1.28)$$

³From now on, we consider here the matrix element of $e^{\phi_I T_0}$, instead of g , in order to exclude the trivial χ_I -dependence of F_I , and, analogously, the ψ - and χ -dependencies of F_G below.

Analogously, in the Gauss case, we get from (1.6) and (1.25) the equation with Hamiltonian (1.26)

$$\begin{aligned} 2j(j+1)F_G^{(j)} &\equiv 2j(j+1) \langle\psi_L|e^{\phi T_0}|\psi_R\rangle_j = \langle\psi_L|e^{\psi T_-}e^{\phi T_0}e^{\chi T_+}\hat{C}|\psi_R\rangle_j = \\ &= \langle\psi_L|e^{\phi T_0}(2T_-T_+ + T_0 + \frac{1}{2}T_0^2)|\psi_R\rangle_j = \\ &= \left(-2\mu_R\mu_L e^{-2\phi} + \frac{\partial}{\partial \phi} + \frac{1}{2} \frac{\partial^2}{\partial \phi^2} \right) \langle\psi_L|e^{\phi T_0}|\psi_R\rangle_j. \end{aligned} \quad (1.29)$$

1.4 Liouville wave function (LWF) and its asymptotics

1.4.1 Solving Liouville equation

Solution

We are looking for the handbook solution to the Liouville equation

$$\frac{\partial^2 f}{\partial \varphi^2} - \mu^2 e^{2\varphi} f = \lambda^2 f. \quad (1.30)$$

The solution is a cylindric function [35], which we choose to be the Macdonald function because of quantum mechanical boundary conditions (f is to be restricted function at pure imaginary λ)

$$f = K_\lambda(\mu e^\varphi). \quad (1.31)$$

Asymptotics

Now let us investigate the asymptotics of this solution. Function (1.31) can be represented in the form which allows one to distinguish easily two different asymptotical exponentials at large negative values of φ :⁴

⁴Let us emphasize that, in the Liouville theory, the oscillating wave exists only at one

$$K_\lambda(\mu e^\varphi) = -\frac{\pi}{2} \frac{I_\lambda(\mu e^\varphi) - I_{-\lambda}(\mu e^\varphi)}{\sin \pi \lambda} \underset{\varphi \rightarrow -\infty}{\sim} \underset{\varphi \rightarrow -\infty}{\sim} -\frac{\pi}{\sin \pi \lambda} \left\{ \frac{1}{\Gamma(1+\lambda)} \left(\frac{\mu e^\varphi}{2} \right)^\lambda - \frac{1}{\Gamma(1-\lambda)} \left(\frac{\mu e^\varphi}{2} \right)^{-\lambda} \right\}, \quad (1.32)$$

where $I_\lambda(z)$ is the Infeld function (the Bessel function of pure imaginary argument).

Let us derive this asymptotical behaviour using integral representation for the Macdonald function:

$$K_\lambda(\mu e^\varphi) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})} (\mu e^\varphi)^\lambda \int_0^\infty \frac{\cos t \, dt}{(t^2 + \mu^2 e^{2\varphi})^{\lambda + \frac{1}{2}}} \longrightarrow \longrightarrow \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})} (\mu e^\varphi)^\lambda \int_0^\infty t^{-2\lambda-1} \cos t \, dt = -\frac{\pi}{2\Gamma(\lambda+1) \sin \pi \lambda} \left(\frac{\mu}{4} \right)^\lambda e^{\lambda\varphi}. \quad (1.33)$$

Here the Γ -function integral is defined whenever $\lambda \leq 0$. If $\lambda \geq 0$, one can redefine variable $t \rightarrow \mu e^\varphi t$ to get instead of the second line in (1.33) the second exponential in (1.32)

$$\frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})} \left(\frac{2}{\mu e^\varphi} \right)^\lambda \int_0^\infty (t^2 + 1)^{-\lambda - \frac{1}{2}} dt = \frac{\Gamma(\lambda)}{2} \left(\frac{2}{\mu} \right)^\lambda e^{-\lambda\varphi}. \quad (1.34)$$

1.4.2 Iwasawa Whittaker LWF

In what follows we reproduce solution (1.31) to the Liouville equation following the group theory line of ss.1.2-1.3 (and also using some direct methods), which is important beyond $SL(2)$ and especially for the affine algebras.

infinity – due to the infinitely increasing potential at the other one. Therefore, we should extract two exponentials $e^{\pm\lambda\varphi}$ at the same asymptotical region. Technically, this is done as follows: one puts λ real and takes it positive and negative respectively to extract appropriate exponentials. Let us remind that the true value of λ is pure imaginary.

To calculate matrix element which solves Liouville equation (1.22), we need the manifest solution to conditions (1.20) in the space of spin- j representation (see sect.1.2.2). We use formulas (1.16). The first condition in (1.20)

$$T_+ \psi_R = \frac{\partial}{\partial x} \psi_R(x) = i\mu \psi_R(x) \quad (1.35)$$

has the evident solution

$$\psi_R(x) = e^{i\mu x}. \quad (1.36)$$

Writing down the second condition, one needs to take into account that the differential operator from (1.16) acts to the left state. Then, this condition is

$$(T_- - T_+) \psi_L = (2x + (1+x^2) \frac{\partial}{\partial x} + 2jx) \psi_L(x) = 0 \quad (1.37)$$

and the solution to this equation is

$$\psi_L(x) = (1+x^2)^{-j-1}. \quad (1.38)$$

Therefore, we get finally

$$F_I^{(j)}(\phi_I) = \langle \psi_L | e^{\phi_I T_0} | \psi_R \rangle_j = \int_{-\infty}^\infty (1+x^2)^{-j-1} e^{\phi_I (2j-2x \frac{\partial}{\partial x})} e^{i\mu x} dx = = e^{2j\phi_I} \int_{-\infty}^\infty (1+x^2)^{-j-1} e^{i\mu x e^{-2\phi_I}} dx \sim e^{-\phi_I} K_{j+\frac{1}{2}}(\mu e^{-2\phi_I}). \quad (1.39)$$

Since the solution to equation (1.22) is $e^{\phi_I} F_I^{(j)}(\phi_I) = K_{j+\frac{1}{2}}(\mu e^{-2\phi_I})$, it indeed coincides with (1.31).

In the course of the calculation, we used in (1.37) the action of the generators T_+ and T_- to the left state, i.e. the corresponding x -derivatives acted to the left. The other way of doing is to act by the same generators to the right state.

It gives the equation

$$(T_+ - T_-) \psi_L = ((1+x^2) \frac{\partial}{\partial x} - 2jx) \psi_L'(x) = 0, \quad (1.40)$$

the solution being

$$\psi'_L(x) = (1+x^2)^j. \quad (1.41)$$

Now, since $\lambda = j + \frac{1}{2}$ is pure imaginary, one can write

$$\psi_L(x) = \overline{\psi'_L(x)}, \quad \text{i.e.} \quad \psi_{L,\lambda}(x) = \psi'_{L,\bar{\lambda}}(x) = \psi'_{L,-\lambda}(x). \quad (1.42)$$

Such a way of doing is correct in general, since the corresponding integral (scalar product) exists because of unitarity of the representation (i.e. pure imaginary λ – see sect.3.2 and Part II).

1.4.3 Gauss Whittaker LWF

Solution

Similarly to the previous subsection, conditions (1.25) have the form in the Gauss case:

$$\begin{aligned} T_+|\psi_R\rangle &= \frac{\partial}{\partial x}\psi_R(x) = i\mu_R\psi_R(x), \\ \langle\psi_L|T_- &= (2x+x^2\frac{\partial}{\partial x}+2jx)\psi_L(x) = i\mu_L\psi_L(x). \end{aligned} \quad (1.43)$$

The solutions to these conditions are

$$\psi_R(x) = e^{i\mu_R x}, \quad \psi_L(x) = x^{-2(j+1)}e^{-\frac{i\mu_L}{x}}. \quad (1.44)$$

Therefore, we get the solution to equation (1.27)

$$\begin{aligned} e^\phi F_G^{(j)}(\phi) &= e^\phi \langle\psi_L|e^{\phi T_0}|\psi_R\rangle_{j=} \\ &= e^\phi \int x^{-2(j+1)}e^{-\frac{i\mu_L}{x}}e^{\phi(2j-2x\frac{\partial}{\partial x})}e^{i\mu_R x} dx = \\ &= \left(\frac{i}{\mu_R}\right)^{-(2j+1)} e^{-(2j+1)\phi} \int_0^\infty x^{-2(j+1)}e^{-\frac{\mu_L\mu_R e^{-2\phi}}{x}-x} dx = \\ &= 2\left(i\sqrt{\frac{\mu_L}{\mu_R}}\right)^{-(2j+1)} K_{2j+1}(2\sqrt{\mu_L\mu_R}e^{-\phi}) dx. \end{aligned} \quad (1.45)$$

At the last stage, we have used some different integral representation (A.2) for the Macdonald function. Let us stress that this expression is a solution to Liouville Schrödinger equation (1.27) for any choices of integration domain, say, for $\int_{-\infty}^{+\infty} dx$ or $\int_0^\infty dx$. The integral converges for the second choice. Then, the change of variable, $x = e^\xi$, is admissible, which brings it to the form of the 0 + 0-dimensional sine-Gordon model:

$$\int_{-\infty}^{+\infty} d\xi \exp\left(-(2j+1)\xi - \mu_L e^{-\xi} - \mu_R e^{\xi-2\phi}\right). \quad (1.46)$$

Asymptotics

For future use, we derive here the asymptotics of the Macdonald function starting with this different integral representation (A.2):

$$K_\nu(z) = \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t-\frac{z^2}{4t}} t^{-\nu-1} dt. \quad (1.47)$$

This integral can be dealt with in complete analogy with (1.33)-(1.34). If $\nu \leq 0$, the term $\frac{z^2}{4t}$ can be thrown away in asymptotics (i.e. at small z) and the integral is equal to

$$K_\nu(z) \underset{z \rightarrow 0}{\sim} \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^\infty e^{-t} t^{-\nu-1} dt = -\frac{\pi}{2 \sin \pi \nu \Gamma(1+\nu)} \left(\frac{z}{2}\right)^\nu. \quad (1.48)$$

On the other hand, if $\nu \geq 0$, one should do the replace $t \rightarrow z^2 t$, and the result is

$$K_\nu(z) \underset{z \rightarrow 0}{\sim} \frac{1}{2} (2z)^{-\nu} \int_0^\infty e^{-\frac{1}{4t}} t^{-\nu-1} dt = \frac{\pi}{2 \sin \pi \nu \Gamma(1-\nu)} \left(\frac{z}{2}\right)^{-\nu}. \quad (1.49)$$

Both these results coincide with formula (1.32).

1.4.4 Liouville solution by the Fourier transform

Here we show how the Liouville equation can be solved immediately using the Fourier transform (see also [23]). This method looks the most direct and trans-

parent in all applications, although we have not worked out it yet in full detail for rank > 2 group.

To begin with, we propose the Fourier transform (indeed, with our definition in this subsection, this is the inverse Fourier transform, which makes intermediate formulas more observable) of Liouville equation (1.30). Then, the equation takes the form

$$\left(-\frac{1}{4}p^2 - e^{2x}\right)\bar{f}(p) = \lambda^2\bar{f}(p), \quad f(x) = \int_{-\infty}^{\infty} dp \exp\{ipx\}\bar{f}(p) \quad (1.50)$$

where, for the sake of simplicity, we put $\mu = 1$ and rescale the coefficient of the first term to be $\frac{1}{4}$. Now there are two ways to solve this equation. The most immediate one is to rewrite the operator e^{2x} as a difference operator and to solve the difference equation obtained. Indeed, this operator acts on functions of p as shift operator $e^{2x}\bar{f}(p) = \bar{f}(p + 2i)$. Therefore, equation (1.50) can be rewritten in the difference form

$$\bar{f}(p + 2i) = -(\lambda^2 + \frac{p^2}{4})\bar{f}(p). \quad (1.51)$$

Its obvious solution is

$$\bar{f}(p) = \Gamma(\frac{p}{2i} + \lambda)\Gamma(\frac{p}{2i} - \lambda). \quad (1.52)$$

This solution can be multiplied by a periodic function still satisfying equation (1.51)⁵. However, the choice of Macdonald solution to equation (1.30) above is consistent with the purely Γ -function solution. Making the inverse Fourier

⁵This is amazing that, after the inverse Fourier transform, the arbitrary periodic function results into only two linearly independent solutions, i.e. the transformation has a huge kernel.

transform, one obtains

$$\begin{aligned} f(\varphi) &= \int dp e^{ip\varphi} \Gamma(\frac{p}{2i} + \lambda) \Gamma(\frac{p}{2i} - \lambda) = \\ &= \int_0^\infty dx \int_0^\infty d\tilde{x} \int_{-\infty}^\infty dp e^{-x-\tilde{x}} x^{\frac{p}{2i}-\lambda-1} \tilde{x}^{\frac{p}{2i}+\lambda-1} e^{ip\varphi} = \\ &= \int_0^\infty \frac{dx}{x} \int_0^\infty \frac{d\tilde{x}}{\tilde{x}} \left\{ \int_{-\infty}^\infty dp (x\tilde{x}e^{-2\varphi})^{\frac{p}{2i}} \right\} e^{-x-\tilde{x}} \left(\frac{\tilde{x}}{x}\right)^\lambda \sim \\ &\sim e^{2\lambda\varphi} \int_0^\infty dx x^{-2\lambda-1} e^{-x-\frac{e^{2\varphi}}{x}} \sim K_{2\lambda}(2e^\varphi). \end{aligned} \quad (1.53)$$

Integrations are performed here in the indicated order so that the first integration over p leads to the δ -function $\delta(\log(x\tilde{x}) - 2\varphi)$. Let us note that this form of the solution is very convenient to get its asymptotics. Indeed, since the singularities of integral (1.53) lie on the imaginary axis, one can close the integration contour at infinity and calculate the integral using Cauchy theorem. Then, the sum of the pole contribution gives the expansion at small $z = 2e^x$, the leading asymptotics is given by the nearest singularity of the integral. Depending on the sign of λ , these are $p_s = \pm 2i\lambda$. Then, integral (1.53) is nothing but the value in this pole:

$$f(x) = \int dp e^{ipx} \Gamma(\frac{p}{2i} + \lambda) \Gamma(\frac{p}{2i} - \lambda) \underset{z \rightarrow 0}{\sim} \Gamma(\pm 2\lambda) z^{\mp\lambda}. \quad (1.54)$$

1.4.5 Liouville solution by canonical transformation

Close to the considered in the previous subsection is the following way of solving the Liouville equation (see also [25]). Liouville equation (1.50) can be considered as the eigenvalue problem of the Hamiltonian

$$\hat{H} = -\frac{1}{4}\hat{p}^2 - e^{2\varphi}, \quad [\hat{p}, \hat{\varphi}] = i \quad (1.55)$$

so that the wave function $f_\lambda(\varphi)$ satisfies the equation

$$(\hat{H} - \lambda^2)f_\lambda(\varphi) = 0. \quad (1.56)$$

One can make the change of canonical variables $\hat{\varphi} \rightarrow \hat{Q} = e^{\hat{\varphi}}$, $\hat{p} \rightarrow \hat{P} = e^{-\hat{\varphi}}(\hat{p} + 2i\lambda)$, $[\hat{P}, \hat{Q}] = i$. Then, in new variables, the eigenvalue problem simplifies

$$\begin{aligned} \hat{H} - \lambda^2 &= -\frac{1}{4}e^{\hat{\varphi}}(e^{-\hat{\varphi}}(\hat{p} + 2i\lambda)^2 - 4i\lambda e^{-\hat{\varphi}}(\hat{p} + 2i\lambda) + 4e^{\hat{\varphi}}) = \\ &= -\frac{1}{4}\hat{Q}\left((\hat{P}^2 + 4)\hat{Q} - i(4\lambda + 1)\hat{P}\right). \end{aligned} \quad (1.57)$$

Now, choosing P -representation, we obtain the non-trivial zero mode of operator (1.57)

$$\bar{f}_\lambda(P) = (P^2 + 4)^{-2\lambda - \frac{1}{2}}. \quad (1.58)$$

Coming back to functions on the original coordinate space (i.e. performing the inverse Fourier transform), we get finally

$$f_\lambda(\varphi) = e^{-2\lambda\varphi} \int_{-\infty}^{\infty} dP (P^2 + 4)^{-2\lambda - \frac{1}{2}} e^{-iPe^\varphi} \sim K_{2\lambda}(2e^\varphi). \quad (1.59)$$

This integral representation coincides with that obtained in the Iwasawa case. Quite analogically, one can use a different canonical transformation to get the Gauss integral representation. Indeed, introducing $\hat{Q} = e^{2\hat{\varphi}}$, $\hat{P} = \frac{1}{2}e^{-2\hat{\varphi}}(\hat{p} + 2i\lambda)$ ($[\hat{P}, \hat{Q}] = i$), one gets

$$\begin{aligned} H - \lambda^2 &= -\frac{1}{4}e^{2\varphi}(e^{-2\varphi}(p + 2i\lambda)^2 - 4i\lambda e^{-2\varphi}(p + 2i\lambda) + 4) = \\ &= -Q(P^2Q - i(2\lambda + 1)P + 1). \end{aligned} \quad (1.60)$$

Now in P -representation the non-trivial zero mode of operator (1.60) is

$$\bar{f}_\lambda(P) = P^{-2\lambda - 1}e^{\frac{i}{P}}, \quad (1.61)$$

and after the inverse Fourier transform, we get finally

$$f_\lambda(\varphi) = e^{2\lambda\varphi} \int dP P^{-2\lambda - 1} e^{-\frac{i}{P} - Pe^{2\varphi}} \sim K_{2\lambda}(2e^\varphi). \quad (1.62)$$

1.4.6 Harish-Chandra functions

Now let us say some words about the above-obtained asymptotics of the LWF. Each separate asymptotics is defined ambiguously, up to overall normalization factor. This means that only the ratio of the coefficients in front of oscillating waves makes an invariant sense. Indeed, this ratio is nothing but reflection S -matrix. However, one can impose some invariant conditions to fix the normalization. Let us require that the asymptotics have no poles at finite momenta (this requirement still fix the normalization ambiguously⁶). For the $SL(2)$ Liouville solutions this means that the asymptotics $\Gamma(\pm\lambda)$ (see (1.39) and (1.45) $-\lambda$ is pure imaginary and is equal to $j + \frac{1}{2}$ for the Iwasawa case and to $2j + 1$ for the Gauss one) do not correspond to the properly normalized LWF but should be additionally multiplied by the function $\sin \pi\lambda$ which eliminates all the poles while bringing additional zeros. The two asymptotics of the so normalized LWF are

$$c_\pm = \frac{1}{\Gamma(1 \pm \lambda)}. \quad (1.63)$$

⁶One way to fix this normalization completely is to require that the leading large- p asymptotics of the Fourier transformed WF does not depend on the energy λ . Say, in the $SL(2)$ case, formula (1.52) implies that this asymptotics is $\log \bar{f}(p) \underset{p \rightarrow \infty}{\sim} \left(\frac{p}{2i} - 1\right) \log \frac{p}{2i} - \frac{p}{i} + o(1)$.

These functions are called Harish-Chandra functions [36] will be discussed for other groups below.

2 $SL(3)$

2.1 Notations

Algebra is completely described by the list of non-vanishing simple root commutation relations

$$\begin{aligned} [T_{+i}, T_{-i}] &= T_{0,i}, \quad [T_{\pm i}, T_{0,i}] = \mp 2T_{\pm i}, \quad i = 1, 2; \\ [T_{\pm 1}, T_{0,2}] &= \pm T_{\pm 1}, \quad [T_{\pm 2}, T_{0,1}] = \pm T_{\pm 2}; \\ [T_{\pm 1}, [T_{\pm 1}, T_{\pm 2}]] &= [T_{\pm 2}, [T_{\pm 2}, T_{\pm 1}]] = 0 \end{aligned} \quad (2.1)$$

and by the commutation relation which defines the third generator $\pm T_{\pm 12} \equiv [T_{\pm 1}, T_{\pm 2}]$ (in order to simplify the notations we sometimes use $\pm T_{\pm 3}$ for this generator). The first fundamental representation:

$$\begin{aligned} T_{0,1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_{0,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ T_{+1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_{+2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_{+12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ T_{-1} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_{-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_{-12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.2)$$

Quadratic Casimir operator:

$$\begin{aligned} C_2 &= \sum_{i=1}^3 (T_{-i} T_{+i} + T_{+i} T_{-i}) + \frac{2}{3} \left(\sum_{i=1}^2 T_{0,i}^2 + T_{0,1} T_{0,2} \right) = \\ &= 2 \sum_{i=1}^3 T_{-i} T_{+i} + 2 \sum_{i=1}^2 T_{0,i} + \frac{2}{3} \left(\sum_{i=1}^2 T_{0,i}^2 + T_{0,1} T_{0,2} \right). \end{aligned} \quad (2.3)$$

In the $SL(3)$ case, we consider only the Gauss decomposition. Iwasawa decomposition in general $SL(N)$ case is considered in sect.3. Then, the parametrization of the group element is

$$\begin{aligned} g_G(\{\psi\}, \{\phi\}, \{\chi\} | T) &= \\ &= e^{\psi_1 T_{-1} + \psi_2 T_{-2} + (\psi_{12} - \psi_1 \psi_2) T_{-12}} e^{\phi_1 T_{0,1} + \phi_2 T_{0,2}} e^{\chi_1 T_{+1} + \chi_2 T_{+2} + (\chi_{12} - \chi_1 \chi_2) T_{12}} = \\ &= e^{\psi_1 T_{-1}} e^{\psi_2 T_{-2}} e^{\psi_{12} T_{-12}} e^{\phi_1 T_{0,1}} e^{\phi_2 T_{0,2}} e^{\chi_1 T_{+1}} e^{\chi_2 T_{+2}} e^{\chi_{12} T_{+12}}. \end{aligned} \quad (2.4)$$

Group element in the fundamental representation:

$$g_G = \begin{pmatrix} 1 & 0 & 0 \\ \psi_1 & 1 & 0 \\ \psi_{12} & \psi_2 & 1 \end{pmatrix} \begin{pmatrix} e^{\phi_1} & 0 & 0 \\ 0 & e^{-\phi_1 + \phi_2} & 0 \\ 0 & 0 & e^{-\phi_2} \end{pmatrix} \begin{pmatrix} 1 & \chi_1 & \chi_{12} \\ 0 & 1 & \chi_2 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.5)$$

Let us also introduce the invariant notations suitable for any group of rank h – simple roots α_i , Cartan matrix $A_{ij} \equiv \alpha_i \cdot \alpha_j$, fundamental weights μ_i which are defined as lying at the dual lattice $\mu_i \cdot \alpha_j = \delta_{ij}$ (i.e. $\mu_i = A_{ij}^{-1} \alpha_j$) and $\rho \equiv \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha = \sum_i \mu_i$, where Δ^+ is the subset of the positive roots. In these notations, $\phi_i = -\mu_i \cdot \phi$ and $C_2 = \sum_{\alpha \in \Delta^+} (T_{-\alpha} T_{+\alpha} + T_{+\alpha} T_{-\alpha}) + \sum_{i=1}^h T_{0,i} A_{ij}^{-1} T_{0,j}$ and $\pm \alpha$ denotes positive and negative roots respectively. For the future needs, we also introduce the roots α_i as vectors in the N -dimensional Cartan plane of $GL(N)$ group: $\alpha_i = e_{i+1} - e_i$, $e_i \cdot e_j = \delta_{ij}$.

2.2 Regular representation

We consider here only the right regular representation since the left one is determined just by the interchange $T_{+i} \leftrightarrow T_{-i}$ and $\psi_i \leftrightarrow \chi_i$. The right regular representation is

$$\begin{aligned}
\mathcal{D}_{+1}^R &= \frac{\partial}{\partial \chi_1}, \quad \mathcal{D}_{+2}^R = \frac{\partial}{\partial \chi_2} + \chi_1 \frac{\partial}{\partial \chi_{12}}, \quad \mathcal{D}_{+12}^R = \frac{\partial}{\partial \chi_{12}}, \\
\mathcal{D}_{0,1}^R &= -2\chi_1 \frac{\partial}{\partial \chi_1} + \chi_2 \frac{\partial}{\partial \chi_2} - \chi_{12} \frac{\partial}{\partial \chi_{12}} + \frac{\partial}{\partial \phi_1}, \\
\mathcal{D}_{0,2}^R &= \chi_1 \frac{\partial}{\partial \chi_1} - 2\chi_2 \frac{\partial}{\partial \chi_2} - \chi_{12} \frac{\partial}{\partial \chi_{12}} + \frac{\partial}{\partial \phi_2}, \\
\mathcal{D}_{-1}^R &= e^{\phi_2 - 2\phi_1} \frac{\partial}{\partial \psi_1} + e^{\phi_2 - 2\phi_1} \psi_2 \frac{\partial}{\partial \psi_{12}} + \\
&+ \chi_1 \frac{\partial}{\partial \phi_1} - \chi_1^2 \frac{\partial}{\partial \chi_1} - \chi_1 \chi_{12} \frac{\partial}{\partial \chi_{12}} - (\chi_{12} - \chi_1 \chi_2) \frac{\partial}{\partial \chi_2}, \\
\mathcal{D}_{-2}^R &= e^{\phi_1 - 2\phi_2} \frac{\partial}{\partial \psi_2} + \chi_2 \frac{\partial}{\partial \phi_2} + \chi_{12} \frac{\partial}{\partial \chi_1} - \chi_2^2 \frac{\partial}{\partial \chi_2}, \\
\mathcal{D}_{-12}^R &= e^{\phi_2 - 2\phi_1} \chi_2 \frac{\partial}{\partial \psi_1} - e^{\phi_1 - 2\phi_2} \chi_1 \frac{\partial}{\partial \psi_2} + (e^{-\phi_1 - \phi_2} + \psi_2 \chi_2 e^{\phi_2 - 2\phi_1}) \frac{\partial}{\partial \psi_{12}} + \\
&+ \chi_{12} \frac{\partial}{\partial \phi_1} - (\chi_1 \chi_2 - \chi_{12}) \frac{\partial}{\partial \phi_2} - \chi_1 \chi_{12} \frac{\partial}{\partial \chi_1} + \chi_2 (\chi_1 \chi_2 - \chi_{12}) \frac{\partial}{\partial \chi_2} - \chi_{12}^2 \frac{\partial}{\partial \chi_{12}}.
\end{aligned} \tag{2.6}$$

The algebra in the highest weight (irreducible) representation induced by the one-dimensional representations of the Borel subalgebra is given by the same formulas (2.6) with all the derivatives $\frac{\partial}{\partial \psi_*}$ vanishing and $\frac{\partial}{\partial \phi_i}$ put equal to j_i ($i = 1, 2$):

$$\begin{aligned}
T_{+1} &= \frac{\partial}{\partial x_1}, \quad T_{+2} = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_{12}}, \quad T_{+12} = \frac{\partial}{\partial x_{12}}, \\
T_{0,1} &= -2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - x_{12} \frac{\partial}{\partial x_{12}} + 2j_1, \\
T_{0,2} &= x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} - x_{12} \frac{\partial}{\partial x_{12}} + 2j_2,
\end{aligned}$$

$$T_{-1} = 2j_1 x_1 - x_1^2 \frac{\partial}{\partial x_1} - x_1 x_{12} \frac{\partial}{\partial x_{12}} - (x_{12} - x_1 x_2) \frac{\partial}{\partial x_2},$$

$$T_{-2} = 2j_2 x_2 + x_{12} \frac{\partial}{\partial x_1} - x_2^2 \frac{\partial}{\partial x_2},$$

$$T_{-12} = 2j_1 x_{12} - 2j_2 (x_1 x_2 - x_{12}) - x_1 x_{12} \frac{\partial}{\partial x_1} + x_2 (x_1 x_2 - x_{12}) \frac{\partial}{\partial x_2} - x_{12}^2 \frac{\partial}{\partial x_{12}}. \tag{2.7}$$

2.3 Hamiltonian – Reducing from regular representation

Casimir operator is (as given in the regular representation):

$$\begin{aligned}
\frac{1}{2}C_2 &= \frac{1}{3} \left(\frac{\partial^2}{\partial \phi_1^2} + \frac{\partial^2}{\partial \phi_2^2} + \frac{\partial^2}{\partial \phi_1 \partial \phi_2} \right) + \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2} + \\
&+ e^{\phi_2 - 2\phi_1} \left(\frac{\partial^2}{\partial \psi_1 \partial \chi_1} + \psi_2 \frac{\partial^2}{\partial \psi_{12} \partial \chi_1} + \chi_2 \frac{\partial^2}{\partial \psi_1 \partial \chi_{12}} + \psi_2 \chi_2 \frac{\partial^2}{\partial \psi_{12} \partial \chi_{12}} \right) + \\
&+ e^{\phi_1 - 2\phi_2} \frac{\partial^2}{\partial \psi_2 \partial \chi_2} + e^{-\phi_1 - \phi_2} \frac{\partial^2}{\partial \psi_{12} \partial \chi_{12}}.
\end{aligned} \tag{2.8}$$

Then, imposing the reduction condition

$$\frac{\partial}{\partial \chi_{12}} F = \frac{\partial}{\partial \psi_{12}} F = 0, \quad \frac{\partial}{\partial \psi_{1,2}} F = i\mu_{1,2}^L F, \quad \frac{\partial}{\partial \chi_{1,2}} F = i\mu_{1,2}^R F, \tag{2.9}$$

i.e.

$$\begin{aligned}
T_{+1,2} |\psi_R\rangle &= i\mu_{1,2}^R |\psi_R\rangle, \quad T_{+12} |\psi_R\rangle = 0, \\
\langle \psi_L | T_{-1,2} &= i\mu_{1,2}^L \langle \psi_L |, \quad \langle \psi_L | T_{-12} = 0,
\end{aligned} \tag{2.10}$$

we obtain the Hamiltonian

$$\begin{aligned}
\frac{1}{2}H &= \frac{1}{3} \left(\frac{\partial^2}{\partial \phi_1^2} + \frac{\partial^2}{\partial \phi_2^2} + \frac{\partial^2}{\partial \phi_1 \partial \phi_2} \right) + \frac{\partial}{\partial \phi_1} + \frac{\partial}{\partial \phi_2} - \\
&- \mu_1^L \mu_1^R e^{\phi_2 - 2\phi_1} - \mu_2^L \mu_2^R e^{\phi_1 - 2\phi_2}.
\end{aligned} \tag{2.11}$$

Then, the scaled matrix element $\Psi(\phi_1, \phi_2) \equiv e^{\phi_1 + \phi_2} F$ satisfies the Liouville equation

$$\begin{aligned} & \left(\frac{1}{3} \left(\frac{\partial^2}{\partial \phi_1^2} + \frac{\partial^2}{\partial \phi_2^2} + \frac{\partial^2}{\partial \phi_1 \partial \phi_2} \right) - \mu_1^L \mu_1^R e^{\phi_2 - 2\phi_1} - \mu_2^L \mu_2^R e^{\phi_1 - 2\phi_2} \right) \Psi(\phi_1, \phi_2) = \\ & = \left(2(j_1 + j_2) + \frac{4}{3}(j_1^2 + j_2^2 + j_1 j_2) + 1 \right) \Psi(\phi_1, \phi_2). \end{aligned} \quad (2.12)$$

Being invariantly written, this equation takes the form

$$\begin{aligned} & \left(A_{ij}^{-1} \frac{\partial^2}{\partial \phi_i \partial \phi_j} - 2 \sum_i \mu_i^L \mu_i^R e^{\alpha_i \cdot \phi} \right) \Psi(\phi_1, \phi_2) = \\ & = (2\mathbf{j} + \boldsymbol{\rho})^2 \Psi(\phi_1, \phi_2) \equiv \boldsymbol{\lambda}^2 \Psi(\phi_1, \phi_2), \end{aligned} \quad (2.13)$$

where $\Psi(\phi_1, \phi_2) \equiv e^{-\boldsymbol{\rho} \cdot \phi} F$, $\mathbf{j} \equiv j_i \boldsymbol{\mu}_i$ and the repeated indices are summed over (eigenvalue of the quadratic Casimir operator in these notations is $2\mathbf{j}(\mathbf{j} + \boldsymbol{\rho})$).

In the Iwasawa case, one can easily obtain the Liouville equation which differs from (2.12) by additional factor 2 in the exponents. This means that its solution can be obtained from the Gauss one by the replace $2\mu_i^L \mu_i^R \rightarrow \mu_i^2$ and $2\mathbf{j} + \boldsymbol{\rho} \rightarrow (\mathbf{j} + \frac{\boldsymbol{\rho}}{2})$, i.e. $2j_i + 1 \rightarrow j_i + \frac{1}{2}$. It is very important that, under this replace, pure imaginary combination remains pure imaginary.

2.4 Liouville wave function (LWF) – Gauss Whittaker LWF

In order to avoid the repetition of clear but tedious calculations, we consider here only Gauss LWF. The Iwasawa $SL(3)$ LWF can be easily obtained from the results for general $SL(N)$ group below.

As before, we can rewrite reduction conditions (2.10) as given on the states

in the highest weight representation. Then, we obtain the equations

$$\frac{\partial}{\partial x_1} \psi_R(x_1, x_2) = i\mu_1^R \psi_R(x_1, x_2), \quad \frac{\partial}{\partial x_2} \psi_R(x_1, x_2) = i\mu_2^R \psi_R(x_1, x_2) \quad (2.14)$$

and

$$\begin{aligned} & \left(x_1^2 \frac{\partial}{\partial x_1} - (x_1 x_2 - x_{12}) \frac{\partial}{\partial x_2} + x_1 x_{12} \frac{\partial}{\partial x_{12}} + 2(j_1 + 1) \right) \psi_L(x_1, x_2, x_{12}) = \\ & = i\mu_1^L \psi_L(x_1, x_2, x_{12}), \\ & \left(-x_{12} \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2} + 2(j_2 + 1)x_2 \right) \psi_L(x_1, x_2, x_{12}) = i\mu_2^L \psi_L(x_1, x_2, x_{12}), \\ & \left(x_1 x_{12} \frac{\partial}{\partial x_1} - x_2(x_1 x_2 - x_{12}) \frac{\partial}{\partial x_2} + x_{12}^2 \frac{\partial}{\partial x_{12}} + \right. \\ & \left. + 2(j_1 + j_2 + 2)x_{12} - 2(j_2 + 1)x_1 x_2 \right) \psi_L(x_1, x_2, x_{12}) = 0. \end{aligned} \quad (2.15)$$

These equations have the following solution:

$$\begin{aligned} & \psi_R(x_1, x_2) = e^{i\mu_1^R x_1 + i\mu_2^R x_2}, \\ & \psi_L(x_1, x_2, x_{12}) = (x_{12} - x_1 x_2)^{-2(j_2 + 1)} x_{12}^{-2(j_1 + 1)} e^{-i\mu_2^L \frac{x_1}{x_{12}} - i\mu_1^L \frac{x_2}{x_1 x_2 - x_{12}}}. \end{aligned} \quad (2.16)$$

Similarly to the $SL(2)$ case, the measure in x -variables is flat, and the integration contour is the real semi-axis. Then, we finally get the LWF as the matrix element

$$\begin{aligned} & \Psi(\phi_1, \phi_2) = e^{\phi_1 + \phi_2} \int dx_1 dx_2 dx_{12} (x_{12} - x_1 x_2)^{-2(j_2 + 1)} x_{12}^{-2(j_1 + 1)} \times \\ & \times e^{-i\mu_2^L \frac{x_1}{x_{12}} - i\mu_1^L \frac{x_2}{x_1 x_2 - x_{12}}} \exp \left\{ \phi_1 \left(-2x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} - x_{12} \frac{\partial}{\partial x_{12}} + 2j_1 \right) + \right. \\ & \left. + \phi_2 \left(x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2} - x_{12} \frac{\partial}{\partial x_{12}} + 2j_2 \right) \right\} e^{i\mu_1^R x_1 + i\mu_2^R x_2} = \end{aligned}$$

$$\begin{aligned}
&= e^{(2j_1+1)\phi_1+(2j_2+1)\phi_2} \int dx_1 dx_2 dx_{12} (x_{12} - x_1 x_2)^{-2(j_2+1)} x_{12}^{-2(j_1+1)} \times \\
&\quad \times e^{-\mu_2^L \frac{x_1}{x_{12}} - \mu_1^L \frac{x_2}{x_1 x_2 - x_{12}} - \mu_1^R x_1 e^{\phi_2 - 2\phi_1} - \mu_2^R x_2 e^{\phi_1 - 2\phi_2}}.
\end{aligned} \tag{2.17}$$

After the change of variables $x_{12} = x_1 x_2 t$, this integral acquires a more invariant form:

$$\begin{aligned}
&e^{(2j_1+1)\phi_1+(2j_2+1)\phi_2} \int dx_1 dx_2 dt (x_1 x_2)^{-2(j_1+j_2+2)+1} (1-t)^{-2(j_2+1)} t^{-2(j_1+1)} \times \\
&\quad \times e^{-\frac{\mu_2^L}{tx_2} - \frac{\mu_1^L}{(1-t)x_1} - \mu_1^R x_1 e^{\phi_2 - 2\phi_1} - \mu_2^R x_2 e^{\phi_1 - 2\phi_2}} \sim \\
&\quad \sim e^{(j_1-j_2)\phi_1+(j_2-j_1)\phi_2} \int dt (1-t)^{-2(j_2+1)} t^{-2(j_1+1)} \times \\
&\quad \times K_{2(j_1+j_2+1)} \left(2\mu_1^L \mu_1^R \frac{e^{\frac{\phi_2}{2} - \phi_1}}{\sqrt{1-t}} \right) K_{2(j_1+j_2+1)} \left(2\mu_2^L \mu_2^R \frac{e^{\frac{\phi_1}{2} - \phi_2}}{\sqrt{t}} \right),
\end{aligned} \tag{2.18}$$

where we used formula (A.2).

2.5 Asymptotics and Harish-Chandra functions

In order to find Harish-Chandra functions, we need to calculate asymptotics of LWF. In the $SL(3)$ case there are 6 different asymptotics, their number being given by the number of elements of the corresponding Weyl group. Let us see how it works. We should consider LWF (2.17) in the asymptotical region where potential is zero, i.e. both exponentials vanish. Still the calculation depends on the signs of $\lambda_i \equiv 2j_i + 1$. The simplest case is when both of them are positive. Then, one can easily get (using formulas of Appendix A)

$$\begin{aligned}
\Psi(\phi_1, \phi_2) &\rightarrow \frac{\pi^3}{\sin \lambda_1 \sin \lambda_2 \sin(\lambda_1 + \lambda_2)} (\mu_1^L \mu_2^L)^{-2(\lambda_1 + \lambda_2)} \times \\
&\quad \times \frac{1}{\Gamma(1 - \lambda_1) \Gamma(1 - \lambda_2) \Gamma(1 - \lambda_1 - \lambda_2)} e^{\lambda_1 \phi_1 + \lambda_2 \phi_2}.
\end{aligned} \tag{2.19}$$

Now let us consider other possible values of λ_i . Totally, there are 6 essentially different domains:

$$\begin{aligned}
&\lambda_1 > 0, \quad \lambda_2 > 0; \\
&\lambda_1 > 0, \quad \lambda_2 < 0, \quad \lambda_1 + \lambda_2 > 0; \\
&\lambda_1 > 0, \quad \lambda_2 < 0, \quad \lambda_1 + \lambda_2 < 0; \\
&\lambda_1 < 0, \quad \lambda_2 > 0, \quad \lambda_1 + \lambda_2 > 0; \\
&\lambda_1 < 0, \quad \lambda_2 > 0, \quad \lambda_1 + \lambda_2 < 0; \\
&\lambda_1 < 0, \quad \lambda_2 < 0.
\end{aligned} \tag{2.20}$$

In each of these domains one should calculate asymptotics of the integral (2.17) redefining properly variables like it has been done in s.1.4.3 for $SL(2)$ case. It is evident that each asymptotics is described by the λ 's lying in the corresponding Weyl chamber, and, therefore, the number of asymptotics is equal to the number of elements of the Weyl group. It is natural to introduce the third $\lambda_3 \equiv \lambda_1 + \lambda_2$. Now, choosing properly the normalization of LWF to cancel all the poles (see s.1.4.6), we finally obtain 6 different Harish-Chandra functions:

$$c_s = \prod_i \frac{1}{\Gamma(1 - s\lambda_i)}, \tag{2.21}$$

where s means Weyl group element and the product goes over all the positive roots (i.e. λ_1, λ_2 and λ_3).

2.6 LWF by the Fourier transform

To complete this section, we consider the solution to the $SL(3)$ Liouville equation in momentum representation. Let us first introduce instead of ϕ_i new

variables $\xi_i \equiv \alpha_i \cdot \phi$. Now, after making the Fourier transform with respect to these variables (i.e. $f(\xi_1, \xi_2) = \int dp_1 dp_2 e^{ip_1 \xi_1 + ip_2 \xi_2} \bar{f}(p_1, p_2)$) and putting $\mu_i^L \mu_i^R = 1$, the Liouville equation takes the following form⁷:

$$\begin{aligned} -\left(p_1^2 + p_2^2 - p_1 p_2 + \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2)\right) \bar{f}(p_1, p_2) = \\ = \bar{f}(p_1 + i, p_2) + \bar{f}(p_1, p_2 + i), \end{aligned} \quad (2.22)$$

where we used that $(\mathbf{j} + \boldsymbol{\rho})^2 = \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2)$. Let us first solve this equation at $\lambda_1 = \lambda_2 = 0$. The solution is

$$\bar{f}(p_1, p_2) = \frac{\Gamma^3\left(\frac{p_1}{i}\right) \Gamma^3\left(\frac{p_2}{i}\right)}{\Gamma\left(\frac{p_1 + p_2}{i}\right)} = \Gamma^2\left(\frac{p_1}{i}\right) B\left(\frac{p_1}{i}, \frac{p_2}{i}\right) \Gamma^2\left(\frac{p_2}{i}\right). \quad (2.23)$$

Now it is already easy to find solutions at any λ_i fitting proper shifts of the Γ -function arguments (let us point out again that in order to get the general solution, one needs to multiply this one by an arbitrary periodic function). The result reads

$$\begin{aligned} \bar{f}(p_1, p_2) = \left[\Gamma\left(\frac{p_1 + p_2}{i}\right)\right]^{-1} \Gamma\left(\frac{p_1}{i} - \left(\frac{2}{3}\lambda_1 + \frac{1}{3}\lambda_2\right)\right) \times \\ \times \Gamma\left(\frac{p_1}{i} + \left(\frac{2}{3}\lambda_2 + \frac{1}{3}\lambda_1\right)\right) \Gamma\left(\frac{p_1}{i} + \left(\frac{1}{3}\lambda_1 - \frac{1}{3}\lambda_2\right)\right) \times \\ \times \Gamma\left(\frac{p_2}{i} + \left(\frac{2}{3}\lambda_1 + \frac{1}{3}\lambda_2\right)\right) \times \\ \times \Gamma\left(\frac{p_2}{i} - \left(\frac{2}{3}\lambda_2 + \frac{1}{3}\lambda_1\right)\right) \Gamma\left(\frac{p_2}{i} - \left(\frac{1}{3}\lambda_1 - \frac{1}{3}\lambda_2\right)\right) = \end{aligned}$$

⁷Note the different signs in the bilinears of p and λ . This is the point which leads to the non-trivial solution to the Fourier Liouville equation.

$$\begin{aligned} = \left[\Gamma\left(\frac{p_1 + p_2}{i}\right)\right]^{-1} \Gamma\left(\frac{p_1}{i} - \bar{\lambda}_1\right) \Gamma\left(\frac{p_1}{i} + \bar{\lambda}_2\right) \Gamma\left(\frac{p_1}{i} + \bar{\lambda}_1 - \bar{\lambda}_2\right) \times \\ \times \Gamma\left(\frac{p_2}{i} + \bar{\lambda}_1\right) \Gamma\left(\frac{p_2}{i} - \bar{\lambda}_2\right) \Gamma\left(\frac{p_2}{i} - \bar{\lambda}_1 + \bar{\lambda}_2\right) = \\ = \Gamma\left(\frac{p_1}{i} - \bar{\lambda}_1\right) \Gamma\left(\frac{p_1}{i} + \bar{\lambda}_2\right) B\left(\frac{p_1}{i} + \bar{\lambda}_1 - \bar{\lambda}_2, \frac{p_2}{i} - \bar{\lambda}_1 + \bar{\lambda}_2\right) \times \\ \times \Gamma\left(\frac{p_2}{i} + \bar{\lambda}_1\right) \Gamma\left(\frac{p_2}{i} - \bar{\lambda}_2\right), \end{aligned} \quad (2.24)$$

where $\bar{\lambda}_1 \equiv \frac{2}{3}\lambda_1 + \frac{1}{3}\lambda_2$, $\bar{\lambda}_2 \equiv \frac{2}{3}\lambda_2 + \frac{1}{3}\lambda_1$ are the projections of the vector $\boldsymbol{\lambda} \equiv \lambda_i \boldsymbol{\mu}_i$ onto the root vectors, i.e. $\bar{\lambda}_i = A_{ij}^{-1} \lambda_j$. The result (2.24) looks as a product over the projections to all 6 vectors. Again one can trivially obtain the asymptotics of this solution looking at the poles of the Γ -functions. The result evidently coincides with (2.19). One can also immediately demonstrate like it has been done for the $SL(2)$ case that the inverse Fourier transform of this solution leads to (2.17). Indeed, one should use the last two lines of (2.24), the second integral representation in (A.1) for the B -function and the integral representation for Γ -functions to prove that

$$\begin{aligned} f(\boldsymbol{\xi}) = \int \dots \int dx_1 d\tilde{x}_1 dx_2 d\tilde{x}_2 dp_1 dp_2 dt x_1^{\frac{p_1}{i} - \bar{\lambda}_1 - 1} \tilde{x}_1^{\frac{p_1}{i} + \bar{\lambda}_2 - 1} x_2^{\frac{p_2}{i} - \bar{\lambda}_2 - 1} \tilde{x}_2^{\frac{p_2}{i} + \bar{\lambda}_1 - 1} (1-t)^{\frac{p_1}{i} + \bar{\lambda}_1 - \bar{\lambda}_2 - 1} t^{\frac{p_2}{i} - \bar{\lambda}_1 + \bar{\lambda}_2 - 1} e^{-x_1 - x_2 - \tilde{x}_1 - \tilde{x}_2} e^{ip_1 \xi_1 + ip_2 \xi_2} = \\ = \int \dots \int \frac{dx_1}{x_1} \frac{d\tilde{x}_1}{\tilde{x}_1} \frac{dx_2}{x_2} \frac{d\tilde{x}_2}{\tilde{x}_2} \frac{dt}{t(1-t)} dp_1 dp_2 (x_1 \tilde{x}_1 (1-t) e^{-\xi_1})^{\frac{p_1}{i}} \times \\ \times (x_2 \tilde{x}_2 t e^{-\xi_2})^{\frac{p_2}{i}} \left(\frac{\tilde{x}_2(1-t)}{x_1 t}\right)^{\bar{\lambda}_1} \left(\frac{\tilde{x}_1 t}{x_2(1-t)}\right)^{\bar{\lambda}_2} e^{-x_1 - \tilde{x}_1 - x_2 - \tilde{x}_2} = \\ = \int \int \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dt}{t(1-t)} \left(\frac{(1-t)e^{\xi_2}}{x_1 x_2 t^2}\right)^{\bar{\lambda}_1} \left(\frac{te^{\xi_1}}{x_1 x_2 (1-t)^2}\right)^{\bar{\lambda}_2} \times \\ \times e^{-x_1 - x_2 - \frac{e^{\xi_1}}{x_1(1-t)} - \frac{e^{\xi_2}}{x_2 t}}, \end{aligned} \quad (2.25)$$

which coincides with (2.18).

3 $SL(N)$

3.1 Notations

Algebra is completely given by the non-zero simple root commutation relations:

$$[T_{\pm i}, T_{0,j}] = \mp A_{ij} T_{\pm i}, \quad [T_{+i}, T_{-j}] = \delta_{ij} T_{0,j}, \quad i, j = 1, \dots, N-1, \quad (3.1)$$

and the Serre relations

$$\text{ad}_{T_{\pm i}}^{1-A_{ij}}(T_{\pm j}) = 0, \quad (3.2)$$

where $\text{ad}_x^k(y) \equiv \underbrace{[x, [x, \dots, [x, y] \dots]]}_{k \text{ times}}$. All other commutation relations can be obtained from (3.1)-(3.2), the generators which corresponds to positive (negative) non-simple roots being constructed from the positive (negative) simple root generators by the manifest formula $[T_{\alpha}, T_{\beta}] = N_{\alpha,\beta} T_{\alpha+\beta}$. Here the generator $T_{\alpha+\beta}$ corresponds to the non-simple root $\alpha + \beta$ and $N_{\alpha,\beta}$ are some non-zero structure constants. With using the non-simple root generators, the Serre identities are replaced by appropriate Lie algebra relations.

Quadratic Casimir operator is

$$C_2 = \sum_{\alpha \in \Delta} T_{\alpha} T_{-\alpha} + \sum_{ij}^{N-1} A_{ij}^{-1} T_{0,i} T_{0,j}, \quad (3.3)$$

where the first sum goes over all (positive and negative) roots.

3.2 Representations

In the case of generic $SL(N)$ group, one can define the (right) regular representation only in general terms of the group acting on the space of the algebra of

functions:

$$\pi_{reg}(h)f(g) = f(gh). \quad (3.4)$$

Therefore, we use from now on mostly group (not algebra) terms. Still, we can restrict the space of functions to the irreducible representations in the generic situation. For doing this, we consider the representation induced by one-dimensional representations of the Borel subgroup. That is, we reduce the space of all functions to the functions satisfying the following covariance property:

$$f_{\lambda}(bg) = \chi_{\lambda}(b)f_{\lambda}(g), \quad (3.5)$$

where b is an element of the Borel subgroup of lower-triangle matrices and χ_{λ} is the character of the Borel subgroup of the form:

$$\chi_{\lambda}(b) = \prod_{i=1}^{N-1} |b_{ii}|^{(\lambda - \rho)e_i} (\text{sign} b_{ii})^{\epsilon_i}, \quad (3.6)$$

where ϵ_i are equal to either 0 or 1. For the sake of simplicity, we consider the representations with all these sign factors to be zero although other cases can be also easily treated. The representation constructed belongs to the principal (spherical) series.

Thus, our representation is given by restricting the space of functions to the functions defined on the coset $B \backslash G$ which, in turn, may be identified with the strictly upper-triangular matrices N_+ . At given λ , there is a natural Hermitian bilinear form on the space of matrix elements of X which is given just by the flat measure:

$$\langle f_L | f_R \rangle_{\lambda} = \int_{X=B \backslash G} \overline{f_{L,\lambda}(x)} f_{R,\lambda}(x) \prod_{ij} dx_{ij}. \quad (3.7)$$

This form becomes a scalar product provided by the pure imaginary λ 's. This gives us unitary irreducible representations of the main (spherical) series.

Now let us describe the structure of the fundamental representations of $SL(N)$, some formulas being used in sect.3.8.

Let us consider the upper-triangle $N \times N$ -matrix with the unit diagonal,

$$||X||_{ij} = x_{ij}\theta(j-i) \ , \quad x_{ii} \equiv 1 \ . \quad (3.8)$$

Let also

$$\Delta_{i_1, \dots, i_k}(X) \equiv \begin{vmatrix} x_{1, i_1} & \dots & x_{1, i_k} \\ \dots & \dots & \dots \\ x_{k, i_1} & \dots & x_{k, i_k} \end{vmatrix} \quad (3.9)$$

These minors are considered as functions on $SL(N)$. Obviously, they are left invariant with respect to the action of the nilpotent subgroup $N_- \subset SL(N)$:

$$\Delta_{i_1, \dots, i_k}(g_- X) = \Delta_{i_1, \dots, i_k}(X), \quad (3.10)$$

where $g_- \in N_-$. Further, the left multiplication by the diagonal matrix $D = \text{diag}(d_1, \dots, d_N) \in SL(N)$ acts as

$$\Delta_{i_1, \dots, i_k}(DX) = \prod_{j=1}^k d_j \Delta_{i_1, \dots, i_k}(X) \ . \quad (3.11)$$

In other words, this implies that

$$\Delta_{i_1, \dots, i_k}(e^{\sum t_j T_{0,j}} X) = e^{t_k} \Delta_{i_1, \dots, i_k}(X) \ . \quad (3.12)$$

Thus, in accordance with (3.5), (3.6), all the minors $\Delta_{i_1, \dots, i_k}(X)$ with fixed k belong to the space of the k -th fundamental representation, \mathcal{F}_k . As a particular case, one can consider the following minors:

$$\Delta_k(X) \equiv \begin{vmatrix} x_{1, N-k+1} & \dots & x_{1N} \\ \dots & \dots & \dots \\ x_{k, N-k+1} & \dots & x_{kN} \end{vmatrix} \ , \quad k = 1, \dots, N \ . \quad (3.13)$$

These minors are right invariant with respect to the action of N_- :

$$\Delta_k(X e^{t T_{-,i}}) = \Delta_k(X) \ , \quad (3.14)$$

where $T_{-,i}$ are lowering generators of $SL(N)$ associated with the simple roots; therefore, $\Delta_k \in \mathcal{F}_k$ is the lowest weight vector of the k -th fundamental representation. Obviously, the weight of Δ_k is determined from the formula

$$\Delta_k(X e^{\sum t_j T_{0,j}}) = e^{-t_{N-k}} \Delta_k(X) \ . \quad (3.15)$$

3.3 Hamiltonians and Liouville equation

In this subsection we derive Liouville equation for $SL(N)$ by the immediate matrix element calculation (cf. 1.3.2). Namely, we impose the following reduction conditions:

In the Iwasawa case:

$$T_{+,i}|\psi_R\rangle = i\mu_i|\psi_R\rangle \quad (3.16)$$

and

$$\langle \psi_L | T_{+,i} = \langle \psi_L | T_{-,i}, \quad \text{i.e.} \quad \langle \psi_L | k = 0, \quad (3.17)$$

where k denotes any algebraic element of the maximal compact subgroup K of $SL(N)$.

In the Gauss case:

$$T_{+,i}|\psi_R\rangle = i\mu_i^R|\psi_R\rangle \quad (3.18)$$

and

$$< \psi_L | T_{-,i} = i\mu_i^L < \psi_L |. \quad (3.19)$$

These constraints are given only by the simple root generators, since all the rest are generated by the commutation relations. Say, the action of non-simple roots generators cancels both the left and right states in the Gauss case and the right state in the Iwasawa case etc.

Now, in the Iwasawa case, one obtains

$$\begin{aligned} (\lambda^2 - \rho^2)F_I^{(\lambda)}(\phi_I) &\equiv (\lambda^2 - \rho^2) < \psi_L | e^{-\mu_i \phi_I T_{0,i}} | \psi_R > = \\ &= < \psi_L | e^{-\mu_i \phi_I T_{0,i}} C_2 | \psi_R > = < \psi_L | e^{-\mu_i \phi_I T_{0,i}} \left(2 \sum_{\alpha \in \Delta^+} T_\alpha T_{-\alpha} + \right. \\ &\quad \left. + 2 \sum_{ij} A_{ij}^{-1} T_{0,j} + \sum_{ij} A_{ij}^{-1} T_{0,i} T_{0,j} \right) | \psi_R > = \\ &= \left(\frac{\partial^2}{\partial \phi_I^2} + 2 \sum_i \frac{\partial}{\partial (\alpha_i \phi_I)} - 2 \sum_i \mu_i^2 e^{2\alpha_i \phi_I} \right) < \psi_L | e^{-\mu_i \phi_I T_{0,i}} | \psi_R >. \end{aligned} \quad (3.20)$$

Therefore, the combination $\Psi_I^{(\lambda)}(\phi_I) e^{-\rho \phi_I} F_I^{(\lambda)}(\phi_I)$ satisfies the following Liouville equation

$$\left(\frac{\partial^2}{\partial \phi_I^2} - 2 \sum_i \mu_i^2 e^{2\alpha_i \phi_I} \right) \Psi_I^{(\lambda)}(\phi_I) = \lambda^2 \Psi_I^{(\lambda)}(\phi_I). \quad (3.21)$$

Similarly, in the Gauss case, one gets

$$\begin{aligned} (\lambda^2 - \rho^2)F_G^{(\lambda)}(\phi) &\equiv (\lambda^2 - \rho^2) < \psi_L | e^{-\mu_i \phi T_{0,i}} | \psi_R > = < \psi_L | e^{-\mu_i \phi T_{0,i}} C_2 | \psi_R > = \\ &= < \psi_L | e^{-\mu_i \phi T_{0,i}} \left(2 \sum_{\alpha \in \Delta^+} T_\alpha T_{-\alpha} + 2 \sum_{ij} A_{ij}^{-1} T_{0,j} + \sum_{ij} A_{ij}^{-1} T_{0,i} T_{0,j} \right) | \psi_R > = \\ &= \left(\frac{\partial^2}{\partial \phi^2} + 2 \sum_i \frac{\partial}{\partial (\alpha_i \phi)} - 2 \sum_i \mu_i^L \mu_i^R e^{\alpha_i \phi} \right) < \psi_L | e^{-\mu_i \phi T_{0,i}} | \psi_R >. \end{aligned} \quad (3.22)$$

and $\Psi_G^{(\lambda)}(\phi) = e^{-\rho \phi} F_G^{(\lambda)}(\phi)$ satisfies the Liouville equation

$$\left(\frac{\partial^2}{\partial \phi^2} - 2 \sum_i \mu_i^L \mu_i^R e^{\alpha_i \phi} \right) \Psi_G^{(\lambda)}(\phi) = \lambda^2 \Psi_G^{(\lambda)}(\phi). \quad (3.23)$$

3.4 On different solutions to the Liouville equations

Throughout this paper we discuss the solutions to the Liouville equation which arise within the group theory framework. Each equation has only one solution of such a type, while generally there is a lot of solutions to the equation. In this subsection, we are going to discuss what conditions select out this unique solution.

Let us start from the simplest $SL(2)$ case. Then, the Liouville equation (1.30) has two linearly independent solutions. One can choose, say, $K_\lambda(e^\varphi)$ and $I_\lambda(e^\varphi)$. From the viewpoint of integral representations we use in the paper, this means different choices of integration contours. One can choose two linearly independent (closed⁸) contours. However, requiring the solution to be restricted function over the real axis (the standard quantum mechanics boundary conditions), one is left with the only solution $K_\lambda(e^\varphi)$. This choice just corresponds

⁸In the compactified complex plane.

to the group theory calculation, because of unitarity of the representations we consider.

As an illustrative example, let us see how the two linearly independent solutions arises in the course of solving the Liouville equation by the Fourier transform. The general solution is solution (1.52) multiplied by an arbitrary periodic function $\Phi(p)$:

$$\bar{f}(p) = \Gamma\left(\frac{p}{2i} + \lambda\right)\Gamma\left(\frac{p}{2i} - \lambda\right)\Phi(p). \quad (3.24)$$

Such a function can be expanded into the Fourier series:

$$\Phi(p) = \sum_n c_n e^{\pi n p}. \quad (3.25)$$

Now we repeat the procedure of sect.1.4.4. It is easily to see that the result is

$$\begin{aligned} f(\varphi) &= \int dp e^{ip\varphi} \Gamma\left(\frac{p}{2i} + \lambda\right)\Gamma\left(\frac{p}{2i} - \lambda\right)\Phi(p) = \\ &= \sum_n c_n \int_0^\infty dx \int_0^\infty d\tilde{x} \int_{-\infty}^\infty dp e^{-x-\tilde{x}} x^{\frac{p}{2i}-\lambda-1} \tilde{x}^{\frac{p}{2i}+\lambda-1} e^{ip\varphi+\pi p n} \sim \\ &\sim \left(\sum_n c_{2n}\right) K_{2\lambda}(2e^\phi) + \left(\sum_n c_{2n+1}\right) K_{2\lambda}(-2e^\phi). \end{aligned} \quad (3.26)$$

Since $K_{2\lambda}(-z) = e^{-2\pi\lambda i} K_{2\lambda}(z) - i\pi I_{2\lambda}(z)$, the result is the arbitrary linear combination of the two independent solutions to the Liouville equation. That is quite amazing that the restricted solution is the simplest one, that with $\Phi(p) = 1$.

The more serious freedom in choosing the solutions to the Liouville equation arises for the higher rank groups $SL(N)$. In this case, the space of solutions is parametrized by $N - 2$ arbitrary parameters. In order to understand what choice of these parameters corresponds to the group theory approach, one needs to observe that the LWF obtained from the group theory satisfies also the

$N - 2$ additional equations generated by the higher Casimir operators (totally there are right $N - 2$ independent higher Casimir operators). These additional equations, along with the boundary conditions (restricted solutions), uniquely fix the solution.

3.5 Equivalence of the Iwasawa and Gauss Whittaker functions

Now one can use the higher Casimir equations to prove that the Iwasawa and Gauss Whittaker functions always coincide, i.e. they are nothing but different integral representations for the same function. Indeed, one needs only to prove that they satisfy the same Liouville (quadratic Casimir) and higher Casimir equations. Let us illustrate how it works for the first two equations.

In fact, as for the first equation – Liouville equation itself – it is evident from comparing (3.21) and (3.23) that they are related by the replace

$$2\phi_I \rightarrow \phi, \quad \mu_i^2 \rightarrow 2\mu_i^L \mu_i^R, \quad \lambda \rightarrow 2\lambda. \quad (3.27)$$

This which proves the equivalence of the corresponding Whittaker functions in the $SL(2)$ case. To deal with higher equations, we need to define the higher Casimir operators for $SL(N)$ group. For doing this, let us fix some representation ρ of the group (in fact, one needs to fix some representation of the universal enveloping algebra). Then, define L -operator [37]

$$L \equiv \sum_{\alpha \in \Delta} \rho(T_\alpha) \otimes T_{-\alpha} + \sum_i A_{ij}^{-1} \rho(T_i) \otimes T_j. \quad (3.28)$$

Now, the k -th Casimir operator can be defined as

$$C_k \equiv \text{Tr}_\rho L^k, \quad (3.29)$$

where trace is taken over the representation ρ . Since the result does not depend on the choice of the representation ρ , one can take the simplest one. Let us look at the first fundamental representation and $SL(3)$ group and calculate the third Casimir operator. For the sake of space, we do not write down the complete answer, however, the result for the Schrödinger equation obtained by the procedure of s.1.3.2 is

$$\begin{aligned} & \left(\frac{\partial^3}{\partial^2 \xi_1 \partial \xi_2} - \frac{\partial^3}{\partial \xi_2^2 \partial \xi_1} + 2 \frac{\partial^2}{\partial \xi_1^2} - \frac{\partial^2}{\partial \xi_1 \partial \xi_2} + \right. \\ & \left. + 2 \frac{\partial}{\partial \xi_1} + 2\eta_1 + \eta_1 \frac{\partial}{\partial \xi_2} - \eta_2 \frac{\partial}{\partial \xi_1} \right) F^{(\lambda)}(\phi) = \\ & = \left(\frac{\lambda_1^3 - \lambda_2^3 + (\lambda_1 - \lambda_2)^3}{27} + \frac{1}{3}(\lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2) - 1 \right) F^{(\lambda)}(\phi), \end{aligned} \quad (3.30)$$

where $\xi_i \equiv \alpha_i \phi$ and $\eta_i \equiv -\mu_i^2 e^{2\xi_i}$ for the Iwasawa case and $\eta_i \equiv -\mu_i^L \mu_i^R e^{\xi_i}$ for the Gauss one. Using this equation, one obtains the following equation for $\Psi^{(\lambda)}(\phi) = e^{-\rho\phi} F^{(\lambda)}(\phi)$

$$\begin{aligned} & \left(\frac{\partial^3}{\partial^2 \xi_1 \partial \xi_2} - \frac{\partial^3}{\partial \xi_2^2 \partial \xi_1} + \eta_1 \frac{\partial}{\partial \xi_2} - \eta_2 \frac{\partial}{\partial \xi_1} \right) \Psi^{(\lambda)}(\phi) = \\ & = \frac{\lambda_1^3 - \lambda_2^3 + (\lambda_1 - \lambda_2)^3}{27} \Psi^{(\lambda)}(\phi), \end{aligned} \quad (3.31)$$

where we used Liouville equations (3.21) and (3.23). Equation (3.31) is the only additional equation arising in the $SL(3)$ case to fix the solution to the Liouville equation (one arbitrary function), and this equation for the Gauss case is again obtained from the Iwasawa case equation by the same replace (3.27). This proves the equivalence of the Iwasawa and Gauss Whittaker functions in the $SL(3)$ case.

3.6 Iwasawa Whittaker LWF

In order to get LWF in the Iwasawa case, as above one needs to construct the solution to the conditions which describes two functions $f_{L,\lambda}(g)$ and $f_{R,\lambda}(g)$ defining left and right states respectively:

$$\begin{aligned} \pi_\lambda(z) f_{R,\lambda}(g) &= f_{R,\lambda}(gz) = e^{\sum i\mu_i z_{i,i+1}} f_{R,\lambda}(g) = e^{i\text{Tr}(\mu z)} f_{R,\lambda}(g), \\ (\mu)_{ij} &= \delta_{i-1,j} \mu_i, \quad g, z \in N_+, \end{aligned} \quad (3.32)$$

and

$$\pi_\lambda(k) f_{L,\lambda}(g) = f_{L,\lambda}(g), \quad k \in K, \quad g \in N_+, \quad (3.33)$$

the algebraic versions of these relations being (3.19) and (3.16) (in the first relation we used formula (3.4)). To solve the first condition is the same as to find a one-dimensional representation of the group of upper-triangular matrices. First, we construct the additive character of this group using the fact that in the product of two upper-triangular matrices, their elements next to the main diagonal are summed. Then, we exponentiate this character to get finally the one-dimensional representation:

$$f_{R,\lambda}^\mu(x) = e^{i\text{Tr}\mu x}. \quad (3.34)$$

The second condition can be solved along the following line. Let us consider the Iwasawa expansion of an element $x \in N_+$:

$$x = n_- \cdot h \cdot k, \quad (3.35)$$

where n_- is an element of the maximal nilpotent subgroup N_- , h is an element of the Cartan subgroup H and k is an element of the maximal compact subgroup K . Then,

$$\begin{aligned}
f_{L,\lambda}(x) &= f_{L,\lambda}(n_-hk) = \chi_\lambda(n_-h)f_{L,\lambda}(k) = \\
&= \chi_\lambda(n_-h) = \chi_\lambda(h) = h_i^{\lambda_i-1} \equiv h^{(\lambda-\rho)e_i}.
\end{aligned} \tag{3.36}$$

The first equality follows from (3.35), and the second one does from (3.33). Now we need to express h_i through matrix elements of x . Indeed, let us make use of (3.35) and obtain the expression for the symmetric matrix xx^t , where index t means transposed matrix:

$$xx^t = n_+h^2n_+^t. \tag{3.37}$$

Now, denoting $\Delta_i(xx^t)$ the upper principal minors of the matrix xx^t , one can easily check that⁹

$$h_i^2 = \frac{\Delta_i}{\Delta_{i-1}}, \quad h_1^2 = \Delta_1, \quad \Delta_N = \frac{\Delta_1}{\Delta_{N-1}}. \tag{3.38}$$

Thus, we have manifestly calculated $f_R(x)$ ((3.34)) and $f_L(x)$ ((3.38)). In order to obtain LWF, we now only need to fix the action of the Cartan part of the group element g on $f_R(x)$. Let us note that, although the element xh with $x \in X$ and $h \in H$ does not belong to X , the element $h^{-1}xh$ does. Therefore, using (3.4), one can get

$$\pi_\lambda(h)f_{R,\lambda}(x) = f_{R,\lambda}(hh^{-1}xh) = h^{\lambda-\rho}f_{R,\lambda}(h^{-1}xh). \tag{3.39}$$

Now, using this formula and manifest expression (3.34), one finally gets¹⁰

⁹To derive this expression, one needs note that the diagonal element g_{ii} of the matrix g depends only on the matrix elements lying in the $i \times i$ submatrix in the left upper corner.

Then, one may use the induction over the rang of the matrix to obtain formula (3.38).

¹⁰In the following two formulas we do not write down explicitly the subscript I of the Cartan torus coordinates ϕ for more transparent formulas.

$$\begin{aligned}
\pi_\lambda(e^{-\mu_i\phi T_{0,i}})f_{R,\lambda}(x) &= e^{\{-\mu_i\cdot\phi\}\{(\lambda-\rho)\cdot(e_{i+1}-e_i)\}}f_{R,\lambda}(e^{\mu_i\phi T_{0,i}}xe^{-\mu_i\phi T_{0,i}}) = \\
&= e^{(\rho-\lambda)\phi}f_{R,\lambda}(e^{\mu_i\phi T_{0,i}}xe^{-\mu_i\phi T_{0,i}}) = e^{(\rho-\lambda)\phi}e^{i\text{Tr}x\mu e^{\alpha\phi}},
\end{aligned} \tag{3.40}$$

where the combination $\mu e^{\alpha\phi}$ means matrix with the matrix elements $\delta_{i-1,j}\mu_i e^{\alpha_i\phi}$. Thus, we finally get for LWF

$$\begin{aligned}
\Psi(\phi) &\sim e^{-\rho\phi} \int_{X=B\backslash G} \prod_{i<j} dx_{ij} h^{-(\lambda+\rho)} e^{(\rho-\lambda)\phi} e^{i\text{Tr}x\mu e^{\alpha\phi}} = \\
&= e^{-\lambda\phi} \int_{X=B\backslash G} \prod_{i<j} dx_{ij} \prod_{i=1}^{N-1} \Delta_i^{-\frac{1}{2}(\lambda\alpha_i+1)}(xx^t) e^{i\mu_i x_{i,i+1} e^{\alpha_i\phi}}.
\end{aligned} \tag{3.41}$$

In this formula we used that $(\lambda+\rho)(e_{i+1}-e_i) = (\lambda+\rho)\alpha_i = \lambda\alpha_i+1$ and that λ is pure imaginary, and, therefore, $\overline{f_{L,\lambda}} = f_{L,\bar{\lambda}} = f_{L,-\lambda}$.

Comment. Let us note that, in the $SL(3)$ case, we used the notations x_1, x_2 and x_{12} for x_{12}, x_{23} and x_{13} respectively.

3.7 Gauss Whittaker LWF. Group derivation

In order to get Gauss LWF we need to use instead of (3.17) condition (3.19) which gives $f_{L,\lambda}(x)$ (the right vector f_R is certainly the same). In the group form, this condition is much analogous to (3.32) and reads as

$$\pi_\lambda(z^t)f_{L,\lambda}(g) = e^{i\text{Tr}(\mu_L z)}f_{L,\lambda}(g). \tag{3.42}$$

To construct this function we use the inner automorphism of the group $SL(N)$ which maps strictly upper-triangular matrices to strictly lower-triangular ma-

trices. This automorphism can be manifestly described as the matrix:

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \dots & 1 & 0 & 0 & \dots & & \\ 0 & 1 & 0 & \dots & 0 & \dots & 0 \\ 1 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (3.43)$$

i.e.

$$S_{ij} = \delta_{i+j, N+1} \quad (3.44)$$

and

$$(S^{-1}zS)_{ij} = z_{N+1-i, N+1-j}. \quad (3.45)$$

Indeed, we need for the matrix S to be an element of the $SO(N)$ group (in general case we construct it as an element of the Weyl group, see Part II). Therefore, it should be normalized to the unit determinant. It can be done by multiplying it by the factor -1 that does not contribute to formula (3.45). Now let us note that the automorphism S reflects the element $z_{i, i-1}$ into the element $z_{N-i, N+1-i}$ instead of $z_{i-1, i}$ (because of condition (3.19) we are only interested in those elements of z lying on the diagonal nearest to the main one). The situation can be corrected by the suitable reflection of the matrix μ . That is, one needs to introduce the new matrix $\tilde{\mu}_L \equiv \mu_{N-i}^L \delta_{i-1, j}$ so that condition (3.42) can be rewritten as

$$\pi_\lambda(SzS^{-1})f_L(g) = f_L(gSzS^{-1})e^{iTr(\tilde{\mu}_L z)}f_{L, \lambda}(gS^{-1}S), \quad (3.46)$$

i.e.

$$f_L(gS^{-1}z) = e^{iTr(\tilde{\mu}_L z)}f_{L, \lambda}(gS^{-1}). \quad (3.47)$$

From the consideration in the previous subsection we know the solution to this equation, at least, in the upper-triangle matrices:

$$f_L(gS^{-1})|_{B_-=0} = f_L(n_+) = e^{iTr(\tilde{\mu}_L n_+)}, \quad (3.48)$$

where n_+ is the upper-triangle part of gS^{-1} . Since we need to find out the solution to (3.46) for $g = x \in N_+$, some recalculation is needed:

$$f_L(xS^{-1}) = \chi_\lambda(xS^{-1})f_L(n_+) = \chi_\lambda(xS^{-1})e^{iTr(\tilde{\mu}_L n_+)}. \quad (3.49)$$

Now we calculate this function in more explicit terms. First, analogously to formula (3.38), diagonal part of xS^{-1} is given by the ratios of the corresponding minors $h_i = \frac{\Delta_i(xS^{-1})}{\Delta_{i-1}(xS^{-1})}$:

$$\chi_\lambda(xS^{-1}) = \prod_i \Delta_i^{(\lambda \alpha_i - 1)}(xS^{-1}). \quad (3.50)$$

To get the formula for the elements $(n_+)_{p-1, p}$ note that they depend only on the $p \times p$ submatrix in the left upper corner. Let us consider $(p-1, p)$ element of the matrix $b^{-1} \equiv n_+g^{-1}$, $g = xS^{-1}$. It is equal to zero by definition of b and we have the identity:

$$(n_+)_{p-1, p}(g^{-1})_{p, p} + (n_+)_{p-1, p-1}(g^{-1})_{p-1, p} = 0. \quad (3.51)$$

Using the explicit expression for elements of the inverse matrix, we obtain the desired expression:

$$(n_+)_{p-1, p} = \frac{\Delta_{p-1, p}(xS^{-1})}{\Delta_{p-1}(xS^{-1})}, \quad (3.52)$$

where $\Delta_{p-1, p}(xS^{-1})$ is defined as the determinant of the $(p-1) \times (p-1)$ submatrix of xS^{-1} with interchanged $p-1$ and p columns.

Thus, we have manifest expressions for f_L and f_R . Collecting them and making calculations completely analogous to those in formula (3.41) (see also (3.39)), we get the final result for the Gauss Whittaker LWF:

$$\Psi(\phi) = e^{-\lambda\phi} \int_{X=B \setminus G} \prod_{i < j} dx_{ij} \prod_{i=1}^{N-1} \Delta_i^{-(\lambda\alpha_i+1)} (xS^{-1}) \times \\ \times e^{i\mu_i^R x_{i,i+1} e^{\alpha_i\phi} - i\mu_{N-i}^L \frac{\Delta_{i,i+1}(xS^{-1})}{\Delta_i(xS^{-1})}}. \quad (3.53)$$

3.8 Gauss Whittaker LWF. Algebraic derivation

The same expression can be derived also in the "algebraic" way (i.e. solving immediately conditions (3.19) and (3.18)) like it was done for $SL(2)$ and $SL(3)$ cases. We demonstrate this by solving the only non-trivial condition (3.19).

The algebra $SL(N)$ acts on functions as

$$a \cdot f(X) \equiv \left. \frac{d}{dt} f(Xe^a) \right|_{t=0} \quad (3.54)$$

for any generator $a \in SL(N)$.

Let us consider the following functions:

$$\phi_k(X) \equiv \frac{T_{+,k} \cdot \Delta_{N-k}(X)}{\Delta_{N-k}(X)}, \quad k = 1, \dots, N. \quad (3.55)$$

Since the action of $T_{+,i}$ (see definition (3.54)) can be represented as the differentiation

$$T_{+,k} = \sum_{j=1}^i x_{jk} \frac{\partial}{\partial x_{j,k+1}}, \quad (3.56)$$

it is clear that $T_{+,k} \cdot \Delta_k(X)$ is the minor which can be obtained from $\Delta_k(X)$ by the permutation of the k -th and $k-1$ -th columns of the matrix X . It is easy to see that the left action of the whole Borel subgroup $B_- \subset SL(N)$ leaves $\phi_k(X)$ invariant. Now we need to calculate the right action of the subgroup N_- , i.e.

$$\phi_k(Xe^{tT_{-,j}}) \equiv \left[(\Delta_{N-k}(Xe^{tT_{-,j}}))^{-1} \frac{d}{ds} \Delta_{N-k}(Xe^{tT_{-,j}} e^{sT_{+,k}}) \right]_{s=0}. \quad (3.57)$$

The first factor in the r.h.s. of (3.57) is right invariant with respect to the action of N_- ; all we need is to calculate the action of N_- in the second factor. Make use of the formula

$$e^{tT_{-,j}} e^{sT_{+,k}} = \exp\left(-\delta_{jk} T_{0,k} \log(1+st)\right) \times \\ \times \exp\left(s(1+st\delta_{jk})T_{+,k}\right) \cdot \exp\left(\frac{t}{1+st\delta_{jk}} T_{-,j}\right) \quad (3.58)$$

Due to the right invariance of Δ_{N-k} , the exponential containing $T_{-,j}$ is dropped out. Thus,

$$\Delta_{N-k}(Xe^{tT_{-,j}} e^{sT_{+,k}}) = \Delta_{N-k}(Xe^{-st\delta_{jk}H_k} e^{sT_{+,k}}) + O(s^2). \quad (3.59)$$

Therefore, with the help of (3.15), equation (3.57) reduces to

$$\phi_k(Xe^{tT_{-,j}}) = \phi_k(X) + t\delta_{jk} \quad (3.60)$$

and, therefore, the function

$$\Phi(X) \equiv \prod_{k=1}^{N-1} \exp\{i\mu_k \phi_k(X)\} \quad (3.61)$$

satisfies the conditions

$$\Phi(B_- X) = \Phi(X), \quad \Phi(Xe^{t_j T_{-,j}}) = e^{t_j i\mu_j} \Phi(X), \quad j = 1, \dots, N-1. \quad (3.62)$$

Now, using this function, we can construct the function which belongs to the representation induced by the character χ_λ and satisfies the condition (3.19)

$$T_{-,i} \cdot f_\lambda(X) = i\mu_i f_\lambda(X). \quad (3.63)$$

For doing this, we multiply $\Phi(X)$ by the proper degrees of $\Delta_k(X)$. Then, using formulas (3.11) and (3.14), one can finally find the solution to (3.19)

$$f_\lambda(X) \equiv \Phi(X) \prod_{k=1}^{N-1} \left\{ \Delta_k(X) \right\}^{\lambda\alpha_k - 1}, \quad (3.64)$$

which coincides with the function f_L given by (3.49) with taking into account (3.50) and (3.52).

3.9 LWF by the Fourier transform

As we could see in the previous examples, working in the Fourier representation gives a very direct and transparent way. Looking at the results obtained for the $SL(2)$ and $SL(3)$ cases, one could hope that there existed the general formula for arbitrary $SL(N)$ expressed through Γ -functions. However, it turns out to be not the case. Indeed, let us consider the Fourier transformed LWF:

$$f(\phi) \equiv \int \prod d^{N-1}p e^{i\mathbf{p}\phi} \bar{f}(\mathbf{p}), \quad (3.65)$$

where $d^{N-1}p$ is the volume element. The function $\bar{f}(\mathbf{p})$ satisfies the following difference equation (we put again $\mu_i^L \mu_i^R = 1$)

$$-\frac{1}{2}(\mathbf{p}^2 + \lambda^2) \bar{f}(\mathbf{p}) = \sum_i \bar{f}(\mathbf{p} + i\alpha_i). \quad (3.66)$$

To get some impression of what the result for LWF could be, we consider the solution to the simpler equation at $\lambda = 0$ for $SL(4)$ case. Then, the equation is

$$\begin{aligned} (p_1^2 + p_2^2 + p_3^2 - p_1 p_2 - p_2 p_3) \bar{f}(p_1, p_2, p_3) = \\ = \bar{f}(p_1 + i, p_2, p_3) + \bar{f}(p_1, p_2 + i, p_3) + \bar{f}(p_1, p_2, p_3 + i), \end{aligned} \quad (3.67)$$

where $p_i \equiv \mathbf{p}\mu_i$. It is already not easy to solve this equation immediately. However, since we know the integral representation for the LWF (see sect.3.7), we can read off the result from the Fourier transform of formula (3.53). It looks like (compare with (2.25))

$$\begin{aligned} & \int \dots \int dx_1 dx_2 dx_3 dx_{12} dx_{23} dx_{123} \frac{1}{x_{123}} \times \\ & \times \frac{1}{x_{123}x_2 - x_{12}x_{23}} \frac{1}{x_{12}x_3 + x_1x_{23} - x_1x_2x_3 - x_{123}} \times \\ & \times e^{-x_1 e^{\xi_1} - x_2 e^{\xi_2} - x_3 e^{\xi_3} - \frac{x_{12}}{x_{123}} - \frac{x_{123} - x_1 x_{23}}{x_{123}x_2 - x_{12}x_{23}} - \frac{x_{23} - x_2 x_3}{x_{12}x_3 + x_1x_{23} - x_1x_2x_3 - x_{123}}} = \\ & = \int \dots \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \frac{dt_{12} dt_{23} dt_{123}}{t_{123}(t_{123} - t_{12}t_{23})(t_{12} + t_{23} - 1 - t_{123})} \times \\ & \times e^{-x_1 e^{\xi_1} - x_2 e^{\xi_2} - x_3 e^{\xi_3} - \frac{1}{x_3} \frac{t_{12}}{t_{123}} - \frac{1}{x_2} \frac{t_{123} - t_{23}}{t_{123} - t_{12}t_{23}} - \frac{1}{x_1} \frac{t_{23} - 1}{t_{12} + t_{23} - 1 - t_{123}}} = \\ & = \int \dots \int \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \frac{d\tilde{x}_1}{\tilde{x}_1} \frac{d\tilde{x}_2}{\tilde{x}_2} \frac{d\tilde{x}_3}{\tilde{x}_3} \frac{dt_{12} dt_{23} dt_{123}}{t_{123}(t_{123} - t_{12}t_{23})(t_{12} + t_{23} - 1 - t_{123})} \times \\ & \times dp_1 dp_2 dp_3 \left(\frac{x_3 \tilde{x}_3 t_{123} e^{-\xi_3}}{t_{12}} \right)^{\frac{p_3}{i}} \left(\frac{x_2 \tilde{x}_2 (t_{123} - t_{12}t_{23}) e^{-\xi_2}}{t_{123} - t_{23}} \right)^{\frac{p_2}{i}} \times \\ & \times \left(\frac{x_1 \tilde{x}_1 (t_{12} + t_{23} - 1 - t_{123}) e^{-\xi_1}}{t_{23} - 1} \right)^{\frac{p_1}{i}} e^{-x_1 - x_2 - x_3 - \tilde{x}_1 - \tilde{x}_2 - \tilde{x}_3} = \\ & = \int \dots \int \left\{ \left[dx_1 dx_2 dx_3 d\tilde{x}_1 d\tilde{x}_2 d\tilde{x}_3 x_1^{\frac{p_1}{i}-1} x_2^{\frac{p_2}{i}-1} x_3^{\frac{p_3}{i}-1} \tilde{x}_1^{\frac{p_1}{i}-1} \tilde{x}_2^{\frac{p_2}{i}-1} \tilde{x}_3^{\frac{p_3}{i}-1} \times \right. \right. \\ & \times e^{-x_1 - x_2 - x_3 - \tilde{x}_1 - \tilde{x}_2 - \tilde{x}_3} \left. \right] dt_{123} dt_{12} dt_{23} \left[t_{123}^{\frac{p_3}{i}-1} t_{12}^{-\frac{p_3}{i}} (t_{123} - t_{12}t_{23})^{\frac{p_2}{i}-1} \times \right. \\ & \times (t_{123} - t_{23})^{-\frac{p_2}{i}} (t_{12} + t_{23} - t_{123} - 1)^{\frac{p_1}{i}-1} (t_{23} - 1)^{-\frac{p_1}{i}} \left. \right] \left. \right\} \times \\ & \times e^{i\xi_1 p_1 + i\xi_2 p_2 + i\xi_3 p_3} dp_1 dp_2 dp_3 \xrightarrow{\text{Fourier transf.}} \Gamma^2\left(\frac{p_1}{i}\right) \Gamma^2\left(\frac{p_2}{i}\right) \Gamma^2\left(\frac{p_3}{i}\right) F\left(\frac{p_1}{i}, \frac{p_2}{i}, \frac{p_3}{i}\right), \end{aligned} \quad (3.68)$$

where we introduced the new variables $t_{12} \equiv \frac{x_{12}}{x_1 x_2}$, $t_{23} \equiv \frac{x_{23}}{x_2 x_3}$, $t_{123} \equiv \frac{x_{123}}{x_1 x_2 x_3}$ and the integral function

$$\begin{aligned} F(\gamma_1, \gamma_2, \gamma_3) \equiv \int \int \int dt_{123} dt_{12} dt_{23} t_{123}^{\gamma_3-1} t_{12}^{-\gamma_3} \times \\ \times (t_{123} - t_{12}t_{23})^{\gamma_2-1} (t_{123} - t_{23})^{-\gamma_2} (t_{12} + t_{23} - t_{123} - 1)^{\gamma_1-1} (t_{23} - 1)^{-\gamma_1}. \end{aligned} \quad (3.69)$$

One can understand that this is the general structure of the Fourier transformed LWF that, with any simple root i , there can be associated $\Gamma^2(\frac{p_i}{i})$. Unfortunately, there is no further factorization for non-simple roots.

The integral $F(\gamma_1, \gamma_2, \gamma_3)$ can be calculated, the result being the Meyer function [38]

$$\begin{aligned}
F(\gamma_1, \gamma_2, \gamma_3) &= \int dx dy dt x^{\gamma_2-1} (x-1)^{-\gamma_1-\gamma_3+1} \times \\
&\times (1-t)^{\gamma_3-1} t^{-\gamma_3} (1-tx)^{\gamma_1-1} (1-y)^{p+1-1} y^{-\gamma_1} (1-yx)^{\gamma_3-1} = \\
&= \Gamma(1-\gamma_3) \Gamma(\gamma_3) \Gamma(1-\gamma_1) \Gamma(\gamma_1) \times \\
&\times \int dx x^{\gamma_2-1} {}_2F_1(\gamma_1, \gamma_3; 1; x) {}_2F_1(1-\gamma_3, 1-\gamma_1; 1; x) = \\
&= G_{44}^{33} \left(1 \left| \begin{array}{cccc} 1-\gamma_2, & \gamma_3, & \gamma_1, & 1-\gamma_2 \\ 0, & \gamma_1-\gamma_2, & \gamma_3-\gamma_2, & 0 \end{array} \right. \right), \tag{3.70}
\end{aligned}$$

where $x \equiv \frac{t_{123} - t_{12}t_{23}}{t_{123} - t_{23}}$, $t \equiv \frac{t_{12}}{x}$, $y \equiv \frac{1-t_{23}}{x}$, ${}_2F_1(a, b; c; z)$ is the hypergeometric function and we used formulas (A.3), (A.4). This function (at given values of arguments) seems not to reduce to more elementary functions like Γ -functions [35, 39, 38], although the Meyer function is defined to be the Mellin transform of the ratio of products of Γ -functions (see formula (A.5)), i.e.

$$F(\gamma_1, \gamma_2, \gamma_3) \sim \int ds \Gamma(\gamma_1 - \gamma_2 + s) \Gamma(\gamma_3 - \gamma_2 + s) \Gamma(1 - \gamma_3 - s) \Gamma(1 - \gamma_1 - s). \tag{3.71}$$

The other possibility to use the Fourier transform is to do a canonical transformation of variables and, then, proceed the calculations in the momentum space (like it was done in sect.1.4.5). This results to the formulas for the Iwasawa (3.41) and Gauss (3.53) integral representations of LWF.

3.10 Harish-Chandra functions

Now let us discuss Harish-Chandra functions for the $SL(N)$ case. There are two different points. The first point concerns the proper normalization, and the second one is what are the asymptotics themselves. To fix the normalization, one needs to remark that, because of the Weyl invariance of the group theory construction (of the matrix elements), the LWF's are to be Weyl invariant if properly normalized. In the Iwasawa case, this requires a non-trivial normalizing factor, and we will discuss it in the next Part and Appendix B. In the Gauss case, just integral (3.53) turns out to be Weyl invariant, by modulo some trivial power factors of i and μ_j^R .

Let us first look at the $SL(2)$ Gauss LWF (1.45). This integral is invariant with respect to action of the Weyl group element $\lambda \rightarrow -\lambda$ provided $\mu_R = i$. This condition removes the trivial non-invariant factor $\left(\frac{i}{\mu_R}\right)^{-\lambda}$. The invariance can be trivially demonstrated by the change of variable in integral (1.45):

$$x \longrightarrow \frac{\mu_L \mu_R e^{-2\phi}}{x}, \tag{3.72}$$

which leads to the same integral, with λ replaced by $-\lambda$.

Of course, Weyl invariance of the integral can be immediately observed from the Fourier transformed LWF (1.52). The invariance of the $SL(3)$ Gauss LWF can be also understood in the simplest way from the Fourier transformed formula (2.24). However, in the previous section we demonstrated that, in the case of higher rank groups, there would be no so simple formula for the Fourier transformed LWF, and one has to use either change of variables in the integrals, or more sophisticated methods in order to prove Weyl invariance of the LWF. The most effective way to prove it in general case is to construct operators that intertwines between representations π_λ and $\pi_{s\lambda}$, where s is an element of the

Weyl group. Since this way is used in the second Part (for the Iwasawa case), we say here some words on change of variables in the Gauss LWF integrals showing up their Weyl invariance.

Let us consider the $SL(3)$ integral (2.17). Then, there are 6 elements of the Weyl group. The distinguished one is the longest element S ($\lambda_1 \rightarrow -\lambda_2$, $\lambda_2 \rightarrow -\lambda_1$), and its action to the integral can be described by the following change of variables:

$$\begin{cases} x_1 \rightarrow \frac{1}{\mu_2^L \mu_2^R e^{\phi_2 - 2\phi_1}} \frac{x_1}{x_{12}} \\ x_2 \rightarrow \frac{1}{\mu_1^L \mu_1^R e^{\phi_1 - 2\phi_2}} \frac{x_2}{x_1 x_2 - x_{12}} \\ x_{12} \rightarrow \frac{1}{\mu_1^L \mu_2^L \mu_1^R \mu_2^R e^{-\phi_1 - \phi_2}} \frac{1}{x_{12}} \end{cases} \quad (3.73)$$

This can be easily generalized to the $SL(N)$ case (see comment at the end of sect.3.6):

$$x_{ij} \rightarrow e^{-\alpha_{ij}\phi} \frac{\Delta_{i,j}(xS^{-1})}{\Delta_i(xS^{-1})}, \quad (3.74)$$

where α_{ij} denotes the positive root corresponding to x_{ij} and we omitted evident μ -factors ($\mu_{N-i}^L \mu_{N-i}^R$ for $x_{i,i+1}$, $\mu_{N-i}^L \mu_{N-i}^R \mu_{N-i-1}^L \mu_{N-i-1}^R$ for $x_{i,i+2}$ etc.). Unfortunately, the other Weyl elements act less trivially (the element S is evidently distinguished in the Gauss LWF integral (3.53)!). Say, the Weyl transformation $\lambda_1 \rightarrow -\lambda_1$, $\lambda_2 \rightarrow \lambda_1 + \lambda_2$ in the $SL(3)$ case is realized by the quite tedious change of variables

$$\begin{cases} x_1 \rightarrow -\frac{\omega x_{12} + 1}{\omega x_2 x_{12}} (x_{12} - x_1 - x_2) \\ x_2 \rightarrow -\frac{\omega x_{12} - 1}{\omega x_{12} + 1} x_2 \\ x_{12} \rightarrow -\frac{1}{\omega} \frac{x_{12} - x_1 x_2}{x_{12}} \end{cases}, \quad (3.75)$$

where we denoted for brevity $\omega \equiv \mu_1^L \mu_1^R e^{\phi_2 - 2\phi_1}$.

Now one can fix the normalization of the LWF by requiring that, first, it has no poles, and, second, it is Weyl invariant. It still preserves the freedom of multiplying by any Weyl-invariant polynomial. In order to fix normalization entirely, one can ask "the minimal configuration", i.e. the absence of any zeroes not fixed by the first two requirements¹¹ (this slightly resembles the notorious CDD-ambiguities in S -matrix but, unlike them, the ambiguities we discuss here do not influence the physical quantities like S -matrix).

After fixing the normalization, one can calculate the asymptotics of the LWF, i.e. Harish-Chandra functions. Since the number of different asymptotics coincides with the number of elements of the Weyl group, the Harish-Chandra functions are labeled by the Weyl group element, and connected by the action of the Weyl group. In the Iwasawa case, the calculation of the asymptotics has been done in paper [40] (see also the next Part). In the Gauss case, it is done in the similar way, the result being (by modulo inessential μ -factors)

$$c_s(\lambda) = \prod_{\alpha \in \Delta^+} \frac{1}{\Gamma(1 + s\lambda \cdot \alpha)}, \quad (3.76)$$

where s means an element of the Weyl group, and the product runs over all the positive roots. This coincides with the properly normalized Iwasawa formula – see the next Part and Appendix B, how it had to be – see s.3.5. This formula is also to be compared with the Harish-Chandra functions obtained for the $SL(2)$ and $SL(3)$ groups (1.63) and (2.21) (see also comment on the affine case in sect.2).

¹¹The same normalization can be fixed in the very different ways like it was done in footnote 6.

Part II

Construction for arbitrary group

Now we are going to formulate the constructions of the previous Part in the very general form, which though being much similar to that discussed above (and sometimes following just the same line) still makes some sense, since allows one to incorporate the Whittaker WF into more general framework and to establish the connections with zonal spherical functions and Calogero-Sutherland systems. On the other hand, this more general approach also turns out to be useful in the affine case.

1 General approach to arbitrary groups

Let G be a semisimple split Lie group. In particular, all complex groups are split. $SL(n, \mathbf{R})$ is also the split group. Such groups have a Whittaker model of representations, which will be described below. We will work with the real forms. The reason why we use the split forms is that they produce the most non-degenerate interactions of the Liouville type. The real forms are more convenient than the complex ones, since they allow one to represent the generic relations and to draw an analogy with the Calogero-Sutherland systems, which we plan to do. The classical real split groups are $A_{n-1} \rightarrow SL(n, \mathbf{R})$, $B_n \rightarrow SO(n+1, n)$, $C_n \rightarrow Sp(n, \mathbf{R})$, $D_n \rightarrow SO(n, n)$.

We start with some basic facts about the representations of noncompact groups, which can be found in the textbooks [20, 21]. The most of relations will be valid for any real forms, not only for the split ones, unless otherwise is

specified.

1.1 Cartan, Iwasawa and Gauss decompositions

Let \mathcal{G} be a split real semisimple Lie algebra, and σ is the Cartan involution ($\sigma^2 = id$, $\sigma \neq id$). There is the *Cartan decomposition* of \mathcal{G} in two eigensubspaces of σ (the \mathbf{Z}_2 grading):

$$\mathcal{G} = \mathcal{K} + \mathcal{P}, \quad \sigma\mathcal{K} = \mathcal{K}, \quad \sigma\mathcal{P} = -\mathcal{P}.$$

There exists such σ that \mathcal{K} is a maximal compact subalgebra in \mathcal{G} . Let \mathcal{A} be a Cartan subalgebra in \mathcal{P} . The split property of \mathcal{G} means that $r = \dim \mathcal{A} = \text{rank}$ of \mathcal{G} and \mathcal{A} serves as a Cartan subalgebra for \mathcal{G} . There is one real split form for any simple complex Lie algebra $\mathcal{G}_{\mathbf{C}}$. A group is split if its algebra is split.

Let $\{\alpha\} = \Delta$ be the root system in the dual space \mathcal{A}^* . It means that

$$\mathcal{G} = \mathcal{G}_0 \oplus_{\alpha} \mathcal{G}_{\alpha},$$

$$\mathcal{G}_{\alpha} = \{x \in \mathcal{G} \mid \text{ad}_h x = \alpha(h)x, \quad h \in \mathcal{A}\}$$

and $\mathcal{G}_0 = \mathcal{A} + \mathcal{M}$, $\mathcal{G}_0 \cap \mathcal{K} = \mathcal{M}$. $\dim \mathcal{G}_{\alpha} = m_{\alpha}$ is called the multiplicity of the root space \mathcal{G}_{α} . For the split forms $m_{\alpha} = 1$ and $\mathcal{M} = \emptyset$.

The involution acts on the root subspaces as

$$\sigma\mathcal{G}_{\alpha} \mapsto -\mathcal{G}_{-\alpha}.$$

Therefore,

$$\mathcal{K} = \oplus_{\alpha} (\mathcal{G}_{\alpha} - \mathcal{G}_{-\alpha}),$$

and

$$\mathcal{P} = \mathcal{A} + \oplus_{\alpha} (\mathcal{G}_{\alpha} + \mathcal{G}_{-\alpha}).$$

The Killing form $\langle \cdot, \cdot \rangle$ on \mathcal{G} is non-degenerate and positive definite on P . It takes the canonical Euclidean form on \mathcal{A} and

$$\langle \mathcal{G}_\alpha, \mathcal{G}_\beta \rangle = \delta_{\alpha, -\beta}.$$

There are two important facts about the Cartan decomposition:

- i) Generic $x \in \mathcal{P}$ can be "diagonalized" by the adjoint action of K : $x = \text{Ad}_k \phi$, $\phi \in \mathcal{A}$, $k \in K$. The element ϕ is defined up to the action of the Weyl group $W = M'/M$ ($M'(M)$ is the normalizer (the centralizer) of A in K). Therefore, a can be taken lying in the Weyl chamber $\Lambda = A/W$.
- ii) Cartan decomposition can be lifted to the polar decomposition

$$G = PK, \quad P = \exp \mathcal{P}.$$

Here $P = G/K$ is a symmetric space of noncompact type and \exp is one-to-one mapping $\exp : \mathcal{P} \rightarrow P$. Therefore, for generic g

$$g = k_1 a k_2, \quad k_j \in K, \quad \log a \in \Lambda \subset \mathcal{A} \subset \mathcal{P}.$$

We fix some ordering in the dual space \mathcal{A}^* . It simply means choosing the hyperplane in \mathcal{A}^* which does not contain any root. It divides \mathcal{A}^* to the positive and negative parts. Let Δ^+ be a subsystem of the positive roots with respect to this ordering ($\Delta = \Delta^+ \cup \Delta^-$). It allows one to specify the positive Weyl chamber $\Lambda^+ = \{a \in \mathcal{A} \mid \alpha(a) > 0 \forall \alpha \in \Delta^+\}$. The simple roots $\alpha \in \Pi \subset \Delta^+$ generate a basis in Δ^+ ; arbitrary $\alpha \in \Delta^+$ is a sum of the simple roots with non-negative integer coefficients.

Let \mathcal{N} be a nilpotent subalgebra generated by the positive root subspaces:

$$\mathcal{N} = \oplus_{\alpha \in \Delta^+} \mathcal{G}_\alpha.$$

The algebra \mathcal{G} can be represented as the direct sum of its subalgebras

$$\mathcal{G} = \mathcal{K} + \mathcal{A} + \mathcal{N}.$$

It is called the *Iwasawa decomposition*. This decomposition can be lifted to the corresponding group element decomposition

$$G = KAN. \quad (\text{I1})$$

Introduce the vector $\rho \in \mathcal{A}^*$ $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha$. Here $m_\alpha = \dim \mathcal{G}_\alpha$ is the multiplicity of α . For the real split group $m_\alpha = 1$ and we come to the conventional formula for ρ . Since $\text{Ad}_a \mathcal{G}_\alpha = e^{\alpha(\log a)} \mathcal{G}_\alpha$,

$$\text{tr}[\text{Ad}]_a|_{\mathcal{N}} = 2\rho(\log a).$$

For technical reasons we will use also another Iwasawa decomposition as well. Let $\bar{N} = \sigma(N)$. It means that its Lie algebra is generated by the negative root subspaces. Then we can represent the group as

$$G = \bar{N}AK. \quad (\text{I2})$$

We call the element $h_I(g)$ coming from these decomposition $g = v h(g) k$ the horospheric projection. We can define as well the other coordinates of g : $\bar{n}(g)$ and $k(g)$. Note that $k(g)$ is defined up to the left multiplication on M (the centralizer of A in K). In general, we have

$$k(gu) = k(g)u, \quad u \in K, \quad k(gm) = k(g), \quad m \in M,$$

$$h_I(vg) = h_I(g), \quad v \in \bar{N}, \quad (1.1)$$

$$h_I(av) = a, \quad \bar{n}(av) = a v a^{-1}, \quad a \in A, v \in \bar{N}.$$

Generic group elements have also the *Gauss representations*. The Gauss decomposition is

$$G = \bar{N}AN,$$

$$g = vhn = bn, \quad n \in N, \quad h \in A, \quad v \in \bar{N}, \quad b \in B,$$

and B is a Borel subgroup. The Gauss coordinates of g are uniquely defined. We will denote them as $h_G(g), n(g), \bar{n}(g)$. It is clear that

$$h_G(rv) = h_G(yr) = r, \quad n(yr) = \text{Ad}_r^{-1}y, \quad y \in N, \quad r \in A, \quad v \in \bar{N}. \quad (1.2)$$

The generalization of the Gauss decomposition is the Bruhat decomposition, which covers the whole G . It allows one to define the Schubert cell decomposition of the flag variety $B \setminus G$. The set $Y \sim N$ is a cell of maximal dimension in $B \setminus G$. On the other hand, due to the Iwasawa decomposition (1.1) the flag variety has the representation

$$B \setminus G \sim M \setminus K,$$

$$n \rightarrow Mk(n), \quad (1.3)$$

which is the diffeomorphism of \bar{N} onto open dense subset of $M \setminus K$. For $SL(2, \mathbf{C})$, it is the usual stereographic projection $\mathbf{C} \rightarrow \mathbf{CP}^1$. The group G acts on the flag variety as a group of diffeomorphisms

$$xg \mapsto k(xg), \quad x = Mk, \quad xg = \bar{N}AMk(xg), \quad (1.4)$$

$$yg \mapsto n(yg), \quad y \in N, \quad yg = \bar{N}An(yg).$$

Consider now integral formulas on G related to the Iwasawa decomposition (1.1) and to the Gauss decomposition. Since G is semisimple, there exists a measure dg , which is both left and right invariant. To derive dg , note that it can be reconstructed from the invariant metric, which, in its turn, is defined by the Killing form

$$\gamma = \langle g^{-1}\delta g, g^{-1}\delta g \rangle.$$

The invariant measure in local coordinates x_1, \dots, x_d , $d = \dim G$ takes the form

$$\omega = (\det \gamma)^{1/2} dx_1 \dots dx_d.$$

If $g = vak$ (I2), then

$$\langle g^{-1}\delta g, g^{-1}\delta g \rangle =$$

$$\langle a^{-1}\delta a, a^{-1}\delta a \rangle + \langle \delta k k^{-1}, \delta k k^{-1} \rangle + \langle \text{Ad}_a^{-1}v^{-1}\delta v, k^{-1}\delta k \rangle.$$

The non-zero components of the metric in the Iwasawa coordinates take the form

$$\gamma_{i,j} = \delta_{i,j} \text{ for the canonical basis } H_j \text{ in } \mathcal{A},$$

$$\gamma_{i,j} = -\text{Killing metric on } K,$$

$$\gamma_{i,j} = 2 \exp \alpha(\log a), \quad i = \alpha, j = -\alpha.$$

The determinant of the metric is equal

$$\det \gamma_{a,b} = \exp 4\rho(\log a).$$

The measure dg can be normalized in such a way that for $g = vak$ and left-invariant measures dv, da, dk

$$\int_G dg f(g) = \int_{\bar{N} \times A \times K} dv da dk f(vak) e^{2\rho(\log a)}.$$

This relation allows one to define an invariant measure dx on $M \setminus K$ such that $\int_{M \setminus K} dx = 1$ and under the right group action it is transformed as

$$\int_{M \setminus K} f(xg) h_I^{-2\rho}(g) dx = \int_{M \setminus K} f(x) dx, \quad (1.5)$$

where we use the notation $h_I^{-2\rho}(xg) = \exp(-2\rho(\log h_I(xg)))$.

The metric in the Gauss coordinates $g = van$, $v \in \bar{N}$ takes the form

$$\langle g^{-1}\delta g, g^{-1}\delta g \rangle = \langle a^{-1}\delta a, a^{-1}\delta a \rangle + 2 \langle \text{Ad}_a \delta n n^{-1} v^{-1} \delta v \rangle.$$

It amounts

$$\begin{aligned}\gamma_{i,j} &= \delta_{i,j} \text{ in the canonical basis } H_j \text{ in } \mathcal{A}, \\ \gamma_{i,j} &= \exp \alpha(\log a), \quad i = \alpha, j = -\alpha.\end{aligned}\tag{1.6}$$

The determinant of the metric is equal to the determinant in the Iwasawa coordinates

$$\det \gamma_{a,b} = \exp 4\rho(\log a).$$

It leads to the Gauss integral formula

$$\int_G dg f(g) = \int_{\bar{N} \times A \times N} dv da dn f(van) e^{2\rho(\log a)}.$$

The function $h_I^{-2\rho}(n)$ is integrable on N . If the Haar measure on $M \setminus K$ is normalized as above and

$$\int_N dn h_I^{-2\rho}(n) = 1,$$

then

$$\int_{M \setminus K} f(Mk) dk_M = \int_N f(Mk(n)) h_I^{-2\rho}(n) dn.\tag{1.7}$$

1.2 Principle series of irreducible representations

Consider the Borel subalgebra $B = \bar{N}AM$ in G and its character

$$\chi_\nu(vam) = \exp(i\nu - \rho)(\log a), \quad m \in M, \quad v \in \bar{N}, \quad a \in A.^{12}$$

Consider the space of smooth functions on G which satisfies the following relation

$$f(bg) = \chi_\nu(b)f(g), \quad b \in B.$$

¹²In principle, this character can be non-trivial on the centralizer M as well. For our construction, it is sufficient to restrict it onto the Cartan subgroup A only.

The principle series of representations π_ν is defined as

$$\pi_\nu(g)f(x) = f(xg).$$

According to the Borel-Weyl theorem, π_ν is realized in the space Γ_ν of holomorphic sections of the line bundle \mathcal{L}_ν over the flag variety $B \setminus G$. There exists a Hermitian scalar product in Γ_ν for $\nu \in \mathbf{R}$

$$\langle \Psi_L | \Psi_R \rangle = \int_{B \setminus G} \Psi_L \cdot \bar{\Psi}_R.$$

Consider the realization of $B \setminus G$ as $M \setminus K$. Then, taking into account the action of G on $M \setminus K$ (1.4) and to the Iwasawa decomposition (I2),

$$\pi_\nu(g)|\Psi \rangle(x) = h_I^{i\nu - \rho}(xg)|\Psi \rangle(k(xg)).\tag{1.8}$$

Due to (1.5) the representation is unitary. We call this realization the compact one.

Similarly, for the realization on N the same representation takes the form

$$\pi_\nu(g)|\Psi \rangle(x) = h_G^{i\nu - \rho}(xg)|\Psi \rangle(n(xg)).\tag{1.9}$$

This realization will be referred to as the noncompact one.

The action $\pi_\nu(g)$ of G on the left vectors in the space Γ_ν in the both realizations is given by the inverse operator $\langle \Psi_L | \pi_\nu(g^{-1})$, as it should be.

1.3 Casimir and Laplace-Beltrami operators

Consider the second order Casimir operator C_2 in the universal enveloping algebra $U(\mathcal{G})$. In the basis in \mathcal{G} we have introduced $\{H_j, \mathcal{G}_{\pm\alpha}, j = 1 \dots, r, \alpha \in \Delta^+\}$ it takes the form

$$C_2 = \sum_{j=1}^r H_j^2 + \sum_{\alpha \in \Delta^+} \mathcal{G}_\alpha \mathcal{G}_{-\alpha} + \mathcal{G}_{-\alpha} \mathcal{G}_\alpha.$$

Taking into account the commutation relations

$$[\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}] = \alpha_j(H_j)H_j,$$

we rewrite C_2 as

$$C_2 = \sum_{j=1}^r (H_j^2 + 2\rho(H_j)H_j) + 2\mathcal{G}_{-\alpha}\mathcal{G}_\alpha. \quad (1.10)$$

Being restricted to the irreducible representations, the Casimir operators act as scalars. To calculate the value of C_2 acting on Γ_ν we remind that the representation π_ν has the highest weight vector ξ_ν . For example, in the compact realization (1.8) it is a constant function

$$\pi_\nu(\mathcal{G}_\alpha)\xi_\nu = 0, \quad \alpha \in \Delta^+$$

and

$$\pi_\nu(H_j)\xi_\nu = (i\nu - \rho)(H_j)\xi_\nu.$$

Acting by $\pi_\nu(C_2)$ on ξ_ν , we find

$$\pi_\nu(C_2) = -\langle \nu, \nu \rangle - \langle \rho, \rho \rangle.$$

We need the explicit form of C_2 in the regular representation

$$f(x) \mapsto f(xg)$$

in the Iwasawa (I1) and Gauss coordinates. The derivation is based on the observation that the second order Casimir coincides with the Laplace-Beltrami operator B , constructed by means of the Killing metric $\gamma_{i,j}$, $i, j = 1, \dots, \dim G$ on G

$$B = \frac{1}{(\det \gamma)^{1/2}} \partial_i \gamma^{i,j} (\det \gamma)^{1/2} \partial_j, \quad (1.11)$$

where $\gamma_{i,j} \gamma^{j,k} = \delta_i^k$. Note that this metric is both left and right invariant providing the invariance of the Laplace-Beltrami operator. Let us calculate the metric in the Iwasawa coordinates (1.1). If $g = kan$, then

$$\begin{aligned} \langle g^{-1} \delta g, g^{-1} \delta g \rangle &= \langle a^{-1} \delta a, a^{-1} \delta a \rangle + \\ &+ \langle k^{-1} \delta k, k^{-1} \delta k \rangle + \langle \text{Ad}_a \delta n n^{-1}, k^{-1} \delta k \rangle. \end{aligned} \quad (1.12)$$

The non-zero components of the metric in the Iwasawa coordinates takes the form

$$\begin{aligned} \gamma_{i,j} &= \delta_{i,j} \text{ for the canonical basis } H_j \text{ in } \mathcal{A}, \\ \gamma_{i,j} &= -\text{Killing metric on } K, \\ \gamma_{i,j} &= -2 \exp \alpha(\log a), \quad i = \alpha, j = -\alpha. \end{aligned} \quad (1.13)$$

The determinant of the metric is equal

$$\det \gamma_{a,b} = \exp 4\rho(\log a).$$

Let

$$\begin{aligned} a^{-1} \delta a &= \sum_{j=1}^r \phi_j H_j, \quad \delta n n^{-1} = \sum_{\alpha \in \Delta^+} n_\alpha \mathcal{G}_\alpha, \\ k^{-1} \delta k &= \sum_{\alpha \in \Delta^+} k_\alpha (\mathcal{G}_\alpha - \mathcal{G}_{-\alpha}). \end{aligned}$$

Then, substituting (1.13) into (1.11), we find B in these local coordinates

$$B_I = e^{-2\rho(\phi)} \sum_{j=1}^r \partial_{\phi_j} e^{2\rho(\phi)} \partial_{\phi_j} - 2 \sum_{\alpha \in \Delta^+} \left[e^{-\alpha(\phi)} \partial_{n_\alpha} \partial_{k_\alpha} - e^{-2\alpha(\phi)} \partial_{n_\alpha}^2 \right]. \quad (1.14)$$

In the same way, the Gauss coordinates lead to the operator

$$B_G = e^{-2\rho(\phi)} \sum_{j=1}^r \partial_{\phi_j} e^{2\rho(\phi)} \partial_{\phi_j} + \sum_{\alpha \in \Delta^+} e^{-\alpha(\phi)} \partial_{n_\alpha} \partial_{v_\alpha}, \quad (1.15)$$

where $v^{-1}\delta v = \sum_{\alpha \in \Delta^+} v_\alpha \mathcal{G}_{-\alpha}$.

For the completeness, we write down the Laplace-Beltrami operator in the coordinates corresponding to the Cartan decomposition. Since in the decomposition $g = k_1 \exp \phi k_2$ the measure depends only on the radial part ϕ $\det \gamma = \prod_{\alpha \in \Delta^+} \sinh^{m_\alpha} \alpha(\phi) = \delta(\phi)$,

$$B_{Cartan} = \frac{1}{\delta^{1/2}(\phi)} \sum_{j=1}^r \partial_{\phi_j} \delta^{1/2}(\phi) \partial_{\phi_j} + K \text{ dependent "angular" operator.} \quad (1.16)$$

Consider the higher Casimir operators. There is a generalization of the representation of C_2 in the Iwasawa coordinates [36]. Let u be an element from the universal enveloping algebra $U(\mathcal{G})$. Then there exists a unique element $p(u) \in U(\mathcal{A})$ such that

$$u - p(u) \in \mathcal{K}U(\mathcal{G}) + U(\mathcal{G})\mathcal{N}.$$

Define the map $\gamma : U(\mathcal{G}) \rightarrow U(\mathcal{A})$, $\gamma(u) = p(u) - H_\rho(u)$, where H_ρ is defined by the Killing metric on \mathcal{A} $\langle H_\rho, H \rangle = \rho(H)$. The map γ induces a homomorphism of the center $Z(\mathcal{G})$ of $U(\mathcal{G})$ (the algebra of the Casimirs) on the algebra of W -invariant polynomials I^W on \mathcal{A}

$$\gamma : Z(\mathcal{G}) \rightarrow I^W(\mathcal{A}). \quad (1.17)$$

Applying this homomorphism to C_2 (1.10), one obtains $\gamma(C_2) = -\langle H, H \rangle - \langle \rho, \rho \rangle$.

1.4 Spherical vectors and zonal spherical functions

It is important to find out such spaces of the irreducible representations that there exist vectors invariant with respect to the compact subgroup K and co-variant with respect to the nilpotent subgroup \bar{N} .

The representations containing vectors $\langle \Psi^K |, |\Psi^K \rangle$ invariant with respect to K are called the *spherical* ones. The principle series of representations described above are spherical. The matrix elements $\Phi_\nu(g) = \langle \Psi^K | \pi_\nu(g) | \Psi^K \rangle$ are called *zonal spherical functions* (ZSF). It follows from their definition that they depend only on the radial part $r(g)$ in the Cartan decomposition $g = k_1 r(g) k_2$. Moreover, $\phi = \log r(g)$ can be chosen to lie in the positive Weyl chamber $\phi \in \Lambda^+$ and $\Phi_\nu(\phi)$ is W -invariant $\Phi_\nu(s\phi) = \Phi_\nu(\phi)$.

Since this matrix element is defined in the irreducible representation, $\Phi_\nu(\phi)$ is the common eigenfunction of all Casimir operators

$$C\Phi_\nu(g) = \gamma(\nu)\Phi_\nu(g),$$

where $\gamma(\nu)$ is determined by (1.17). In particular, for C_2 , writing it as the Laplace-Beltrami operator in the Cartan coordinates (1.16) and using the K -bi-invariance of the matrix element, we come to the equation

$$\sum_{j=1}^r \left[\partial_j^2 + \sum_{\alpha \in \Delta^+} m_\alpha \coth \alpha(\phi_j) \partial_j \right] \Phi_\nu(\phi) = -(\langle \nu, \nu \rangle + \langle \rho, \rho \rangle) \Phi_\nu(\phi).$$

After the gauge transform

$$\Phi_\nu(\phi) \mapsto \psi_\nu(\phi) = \frac{\Phi_\nu(\phi)}{\delta^{1/2}(\phi)},$$

we come to the eigenvalue problem for the Calogero-Sutherland system [11]

$$\left[-\sum_{j=1}^r \partial_j^2 + \sum_{\alpha \in \Delta^+} \frac{m_\alpha(m_\alpha - 1)}{2 \langle \alpha, \alpha \rangle} \frac{1}{\sinh^2 \alpha(\phi)} \right] \psi_\nu(\phi) = E_\nu \psi_\nu(\phi).$$

Independently, ZSF $\Phi_\nu(\phi)$ can be uniquely fixed by the three properties:

- i) It is a common eigenfunction of the Casimir operators with the eigenvalue $\gamma(\nu)$ (1.17);

- ii) It is a K -bi-invariant function on G , and, therefore, it lives on the double coset space $K \setminus G/K \sim \Lambda$;
- iii) $\Phi_\nu(\phi)|_{\phi=0} = 1$.

It can be proved that the ZSF as the functions of ν are W -invariant

$$\Phi_{s\nu}(\phi) = \Phi_\nu(\phi), \quad s \in W. \quad (1.18)$$

Let us write down explicitly the integral representation for ZSF (the Harish-Chandra formulas [36]), and, thereby, for the wave functions of the Calogero-Sutherland system. The compact case is the simplest one, since the invariant state is just a constant function:

$$\Phi_\nu(g) = \int_{M \setminus K} dx h_I^{i\nu-\rho}(xg).$$

Note that, instead of g , we can use here the element $r(g) = \exp(\phi) \in A$ from the Cartan decomposition $g = k_1 r k_2$.

The noncompact case can be treated similarly, or, equivalently, one can use the generalized stereographic map $Mk \rightarrow v \in N$ (1.3),(1.7). It can be proved that the K -invariant states in the noncompact realization are

$$\langle \Psi_L^K | (y) = h_I^{i\nu-\rho}(y), \quad y \in N.$$

It follows from the equality

$$h_I(n(yu)) = h_I(y)h_G^{-1}(yu), \quad u \in K, \quad y \in N,$$

which can be derived by comparing the Gauss and the Iwasawa decompositions (I2) for y , using (1.1). Taking into account that

$$h_G(yr) = r, \quad n(yr) = r^{-1}yr, \quad r \in A$$

and using (1.9), we come to the expression

$$\Phi_\nu(r) = r^{i\nu-\rho} \int_N dy h_I^{i\nu-\rho}(r^{-1}yr) h_I^{-i\nu-\rho}(y). \quad (1.19)$$

Let

$$D = \{\nu \in \mathcal{A}^* | \Im m \nu_\alpha < 0, \quad \forall \alpha \in \Delta^+\}. \quad (1.20)$$

Consider $r = r(t) = \exp tH$, $H \in \Lambda^+$, $t \rightarrow +\infty$. Since $\text{Ad}_{r(t)}^{-1}y \rightarrow id$ and integral converges for $\nu \in D$ in this limit,

$$\Phi_\nu(r(t)) \sim r^{i\nu-\rho} c(\nu).$$

Here $c(\nu)$ is the Harish-Chandra function

$$c(\nu) = \int_N dy h_I^{-i\nu-\rho}(y). \quad (1.21)$$

This integral was calculated explicitly by the factorization procedure (Gindikin-Karpelevich formula [40]).

$$c(\nu) = \prod_{\alpha \in \Delta^+} c_\alpha(\nu), \quad c_\alpha(\nu) = I_\alpha(i\nu)/I_\alpha(\rho),$$

where for the split groups

$$I_\alpha(\nu) = B\left(\frac{1}{2}, \nu_\alpha\right), \quad \nu_\alpha = \frac{\alpha(\nu)}{\langle \alpha, \alpha \rangle}. \quad (1.22)$$

The Harish-Chandra function defines the scattering in the Calogero-Sutherland model, since for $\log r(t) \in \Lambda^+$, $\nu \in \mathcal{A}^*$, $t \rightarrow +\infty$

$$\Phi_\nu(r(t)) \sim \sum_{s \in W} c(s\nu) r^{si\nu-\rho}. \quad (1.23)$$

The scattering process is reduced to the two particle collisions due to the Gindikin-Karpelevich formula.

2 Whittaker model.

Here we present the general theory of the Whittaker models. Their concrete realizations for particular groups were given in Part I.

Consider one-dimensional representation ψ_μ of the group \tilde{N} :

$$\psi_\mu(v_1 v_2) = \psi_\mu(v_1) \psi_\mu(v_2), \quad \mu \in \mathcal{A}^*.$$

It can be constructed as follows. Represent v as an element of a matrix group \tilde{N} as $v = v' + \tilde{v}$, where $\tilde{v} \in [\tilde{N}, \tilde{N}]$. The element v' can be decomposed as $v' = \sum_{\alpha \in \Pi} v_\alpha$, $v_\alpha \in \mathcal{G}_\alpha$. Since $v_1 v_2 = v'_1 + v'_2 + \tilde{v}$,

$$\psi_\mu(v) = \exp i\mu(v') = \exp i \sum_{\alpha \in \Pi} \mu_\alpha v_\alpha. \quad (2.1)$$

The group N has similar representations.

Consider the space $C_{\nu, \mu}^\infty$ of smooth functions on G , which can be characterized by the following properties:

- i) $V_\nu(g; \mu)$ are common eigenfunctions of the Casimir operators with the eigenvalues $\gamma(\nu)$;
- ii) $V_\nu(kgn; \mu) = \psi_\mu(n) V_\nu(g; \mu)$, $n \in N, k \in K$ (the covariance condition).

This repeats the corresponding conditions for the ZSF. Let $V_\nu(g; \mu) \in C_{\nu, \mu}^\infty$. Then, taking $g = k(\exp \phi)n$, we find from (1.14) that it satisfies the equation

$$\left[e^{-2\rho(\phi)} \sum_{j=1}^r \partial_{\phi_j} e^{2\rho(\phi)} \partial_{\phi_j} - 2 \sum_{\alpha \in \Pi} \mu_\alpha^2 e^{-2\alpha(\phi)} \right] V_\nu(\phi; \mu) = -(\nu^2 + \rho^2) V_\nu(\phi; \mu). \quad (2.2)$$

Making substitution $V_\nu(\phi; \mu) = \exp(-\rho(\phi)) \varphi_\nu(\phi; \mu)$, one obtains

$$\left[\sum_{j=1}^r \partial_{\phi_j}^2 - 2 \sum_{\alpha \in \Pi} \mu_\alpha^2 e^{-2\alpha(\phi)} \right] \varphi_\nu(\phi; \mu) = -\nu^2 \varphi_\nu(\phi; \mu). \quad (2.3)$$

It is just the eigenvalue problem for the Schrödinger operator for the open Toda lattice.

Let L be a root sublattice

$$L = \sum_{\alpha \in \Pi} n_\alpha \alpha, \quad n_\alpha \text{ are even integer and non-negative.}$$

Define rational functions $a_\gamma(\nu)$ on \mathcal{A}^* depending on the vertices γ of L by the recurrence relation

$$(-\nu^2 + 2 < \nu, \gamma >) a_\gamma(\nu) - 2 \sum_{\alpha \in \Pi} \mu_\alpha^2 a_{\gamma-2\alpha}(\nu) = 0, \quad a_0(\nu) = 1.$$

Then, it can be demonstrated by direct calculations that

$$\varphi_\nu(\phi; \mu) = e^{i\nu(\phi)} \sum_{\gamma \in L} a_\gamma(\nu) e^{-\gamma(\phi)}. \quad (2.4)$$

Let $\tilde{\mathcal{A}}^*$ be the complement in \mathcal{A}^* to the union of hyperplanes

$\sigma_\nu = \{-\nu^2 + 2 < \nu, \gamma > = 0\}$. Then, the series converges absolutely and uniformly for $\phi \in \mathcal{A}$, $\nu \in \tilde{\mathcal{A}}^*$ [18]. Moreover, it was proved that

$$V(r; s\nu, \mu), \quad s = id, \dots, s_0, \quad s \in W$$

generate a basis in $C_{\nu, \mu}^\infty(A)$, $\dim C_{\nu, \mu}^\infty(A) = \sharp W$. It is easy to see that they are unbounded on A .

Of our main interest, however, are bounded functions in this space. In fact, there exists only one such a function in $C_{\nu, \mu}(A)$. It is precisely the function we call *the Whittaker function* (WF). Following the same ideology as in the ZSF

case, we construct WF's as matrix elements in the irreducible space Γ_ν . We also add to i), ii) the normalization condition which is not so evident as iii) for the ZSF. We work in the noncompact realization. It follows from the covariant condition that we should define the state $w(y; \mu) = |\Psi_R^\mu >$ such that

$$\pi_\nu(n)|\Psi_R^\mu >(y) = \psi_\mu(n')|\Psi_R^\mu >(y), \quad (n = n' + \tilde{n}, \quad \tilde{n} \in [N, N]).$$

It is clear that $w(y; \mu) = \psi_\mu(y) = \exp i\mu(y')$. This state is called *the Whittaker vector*. There is only one Whittaker vector in Γ_ν up to the constant multiplier. We have already defined the K -invariant states in the noncompact realization: $< \Psi_L^K | = h_I^{i\nu-\rho}(y)$. The matrix element $< \Psi_L^K | \pi_\nu(g) | \Psi_R^\mu >$ satisfies both conditions i) and ii). Thus, using (1.1), we find

$$W_\nu(r; \mu) = r^{-i\nu+\rho} \int_N dy h_I^{i\nu-\rho}(ryr^{-1}) \exp i\mu(y'). \quad (2.5)$$

The integral converges for $\nu \in \mathcal{A}_\mathbb{C}^*$, $\Re \nu \in \Lambda$ [18]. Therefore, we have constructed the WF – the desirable bounded eigenfunction. This expression is close to the similar integral for the ZSF. It can be rewritten in another form, if one considers the group action on the right state. Using (1.2), we obtain

$$W_\nu(r; \mu) = r^{-i\nu-\rho} \int_N dy h_I^{i\nu-\rho}(y) \exp i\mu(r^{-1}yr)'. \quad (2.6)$$

Like the ZSF case assume that $r = r(t) = \exp tH$, $H \in \Lambda^+$, $t \rightarrow +\infty$ $\text{Ad}_{r^{-1}}y = id$. Then, we obtain from (1.21) and (2.6) for $-\nu \in D$ (1.20)

$$\lim_{t \rightarrow \infty} r^{i\nu+\rho} W_\nu(r(t); \mu) = c(-\nu). \quad (2.7)$$

It will be instructive to compare these integral representations with the similar formulas for $G = SL(2, \mathbf{R})$ in Part I. In this case, we have $y = y' \in \mathbf{R}$, $r \in \mathbf{R}^+$

$$h_I(y) = \text{diag} \left((1+y^2)^{\frac{1}{2}}, (1+y^2)^{-\frac{1}{2}} \right), \quad (r^{-1}yr)' = yr^{-2},$$

$$i\nu - \rho = \text{diag} \left(i\nu - \frac{1}{2}, -i\nu + \frac{1}{2} \right).$$

Putting $r = e^\phi$ we come to the integrals (compare with (I.1.39))

$$W_\nu(\phi; \mu) = e^{(-2i\nu+1)\phi} \int_{-\infty}^{\infty} \frac{\exp(-i\mu y)}{(1+y^2 e^{4\phi})^{-i\nu+\frac{1}{2}}} dy,$$

$$W_\nu(\phi; \mu) = e^{(-2i\nu-1)\phi} \int_{-\infty}^{\infty} \frac{\exp(-i\mu y e^{-2\phi})}{(1+y^2)^{-i\nu+\frac{1}{2}}} dy.$$

The direct calculation gives

$$W_\nu(\phi; \mu) = \left(\frac{2}{\mu} \right)^{i\nu} \frac{1}{\Gamma(\frac{1}{2} - i\nu)} e^{-\phi} K_{-i\nu}(\mu e^{-2\phi})$$

as it should be. Now we present the expansion of the WF in the basis $V(r; s\nu, \mu)$, $s \in W$ [18]. It defines the scattering in the model and leads eventually to the relation similar to (1.23) for the ZSF:

$$W_\nu(r; \mu) = \sum_{s \in W} b(s; \nu) V(r; s\nu, \mu). \quad (2.8)$$

In contrast to (1.23) it is the exact formula, although the right hand side is the sum of unbounded functions. The calculation of the coefficients $b(s; \nu)$ is based on the following important property of the WF (compare with (1.18) for the ZSF)

$$W_\nu(r; \mu) = M(s; \nu, \mu) W_{s\nu}(r; \mu), \quad (2.9)$$

where $M(s; \nu, \mu)$ is a meromorphic function of ν which is determined recursively as follows. If s_α , $\alpha \in \Pi$ is a simple reflection, then

$$M(s_\alpha; \nu, \mu) = e_\alpha(\nu) e_\alpha(-\nu)^{-1} \left(\frac{|\mu_\alpha|}{\sqrt{2} < \alpha, \alpha > } \right)^{2\nu_\alpha}, \quad (2.10)$$

$$e_\alpha^{-1}(\nu) = \Gamma \left(\frac{1}{2} - \nu_\alpha \right).$$

Let $l(s)$ be the length of s . It means the minimal number of simple reflections in $s = s_{\alpha_1} \dots s_{\alpha_n}$. If $l(s_\alpha s) = l(s) + 1$, then

$$M(s_\alpha s; \nu, \mu) = M(s; \nu, \mu)M(s_\alpha; \nu, \mu). \quad (2.11)$$

These relations are coming from expressions for the intertwiners for the equivalent representations π_ν and $\pi_{s\nu}$. They were investigated from this point of view in [16].

It can be proved that for $\phi \in \Lambda^+$, $\nu \in D$

$$\lim_{t \rightarrow \infty} V_\nu(e^{\phi t}; \mu) e^{(s\nu + \rho)(\phi t)} = 0,$$

unless $s = id$. Otherwise, this limit is equal to 1. In fact, $\Re(\nu - s\nu)(\phi) < 0$ for $s \neq id$, $\phi \in \Lambda^+$, $-\nu \in D$ (Lemma 3.3.2.1 in [20]). Then, this statement follows from the expansions of $\varphi_\nu(\phi; \mu)$ (2.4). We obtain from (2.7) $b(id; \nu) = c(-\nu)$. On the other hand, it follows from (2.8) and (2.9) that

$$b(s_1 s_2; \nu, \mu) = M(s_2; \nu, \mu) b(s_1; s_2 \nu, \mu),$$

and, in particular,

$$b(s; \nu, \mu) = M(s; \nu, \mu) c(-s\nu). \quad (2.12)$$

Thus, eventually, the analog of (1.23) for the WF takes the form

$$W_\nu(r; \mu) = \sum_{s \in W} M(s; \nu, \mu) c(-s\nu) V(r; s\nu, \mu), \quad (2.13)$$

where $M(s; \nu, \mu)$ is determined by (2.10) and (2.11). We present expansion (2.13) explicitly determining an appropriate normalization. For the rank one case $V(r; s\nu, \mu) = \Gamma(1/2 - i\nu) I_{-i\nu}(\mu e^{-2\phi})$ (see (I.1.32) and (I.1.39)). We can multiply the WF by arbitrary function depending on ν only. It still satisfies i), ii). We demonstrate that after the transformation

$$\tilde{W}_\nu(r; \mu) = \xi(\nu) W_\nu(r; \mu),$$

where

$$\xi(\nu) = \prod_{\alpha \in \Delta^+} \frac{\sin i\pi\nu_\alpha}{\pi} \Gamma(1/2 - i\nu_\alpha), \quad (2.14)$$

all the poles in the expansion of $W'_\nu(r; \mu)$ disappear:

$$\tilde{W}_\nu(r; \mu) = \sum_{s \in W} b'(s; \nu, \mu) V(r; s\nu, \mu),$$

$$b'(s; \nu, \mu) = \prod_{\alpha \in \Delta^+} \frac{\det s}{\Gamma(1 + (-s i\nu)_\alpha)} \left(\frac{|\mu_\alpha|}{\sqrt{2 < \alpha, \alpha >}} \right)^{2 \sum_\gamma i\nu_\gamma}. \quad (2.15)$$

Here \sum_γ is taken over those of the simple roots γ_i which contribute to the representation $s = s_{\gamma_1} \dots s_{\gamma_n}$, and $\det s = \pm 1$, $\det s_\alpha = -1$. The proof of (2.15) is given in Appendix B. Moreover, if we choose $|\mu_\alpha| = \sqrt{2 < \alpha, \alpha >}$, then, as it follows from (2.14), (2.15), $\tilde{W}_\nu(r; \mu)$ becomes W antisymmetric in the ν variables. Thus, this normalization is fixed up to the multiplication by W symmetric polynomial in ν .

Now we define another type of the bounded WF, which we call the Gauss Whittaker functions. It satisfies i) and, instead of ii),

$$\text{ii}^G) \quad W_\nu^G(vgn; \mu^L, \mu^R) = \psi_{\mu^L}(n) \psi_{\mu^R}(v) W_\nu^G(g; \mu^L, \mu^R), \quad v \in \bar{N}, n \in N.$$

It follows from the Gauss decomposition that it also lives in the Cartan algebra \mathcal{A} . In particular, it satisfies the second order differential equation (see (1.15))

$$\left[e^{-2\rho(\phi)} \sum_{j=1}^r \partial_{\phi_j} e^{2\rho(\phi)} \partial_{\phi_j} - \sum_{\alpha \in \Pi} \mu_L \mu_R e^{-\alpha(\phi)} \right] W_\nu^G(\phi; \mu^L, \mu^R) = (-\nu^2 + \rho^2) W_\nu^G(\phi; \mu^L, \mu^R). \quad (2.16)$$

This equation coincides with (2.2) for the Iwasawa WF after the redefinition

$$\nu_I = 2\nu_G, \quad \mu_L \mu_R = 2\mu_I^2, \quad (2.17)$$

where subscribes G and L refer to the Gauss and Iwasawa cases.

We suggest that this two functions after the identification of their parameters coincide up to the multiplication by W symmetric polynomial in ν . To this end, we define $W_\nu^G(g; \mu^L, \mu^R)$ explicitly. We need the state $\langle \Psi_L^{\mu_L} |$ covariant with respect to the $\pi_\nu(\bar{N})$:

$$\langle \Psi_L^{\mu_L} | \pi_\nu(v) = \psi_{\mu_L}(v) \langle \Psi_L^{\mu_L} |, \quad v \in \bar{N}.$$

This state can be read off from the explicit realization of $\pi_\nu(\bar{N})$ (1.9). Let $S \equiv s_0$ be the longest element of the Weyl group. It transforms the set of positive roots into the set of negative roots. We keep the same notation for the representative of $S \in W \sim M'/M$ in $M' \subset K$. Thus, $\text{Ad}_S N = \bar{N}$. Consider the state $\langle \Psi_L^\mu | = \exp i\mu_L(y)$, covariant with respect to the N action. Then, the state $\langle \Psi_L^{\mu_L} | \pi_\nu(S)$ is \bar{N} -covariant

$$\langle \Psi_L^{\mu_L} | \pi_\nu(S) \pi_\nu(v) = \psi_{\mu_L}(v) \langle \Psi_L^{\mu_L} | \pi_\nu(S).$$

Explicitly,

$$\langle \Psi_L^{\mu_L} | \pi_\nu(S) = h_G^{i\nu-\rho}(yS) \exp i\mu_L(\bar{n}'(yS)).$$

It allows one to write down the GLWF:

$$W_\nu^G(r; \mu^L, \mu^R) = r^{-i\nu-\rho} \int_N dy h_G^{i\nu-\rho}(yS) \exp i\{\mu_L(n'(yS)) + \mu_R(r^{-1}yr')\}. \quad (2.18)$$

This integral converges absolutely for $\nu \in \mathcal{A}^*$, $\log r \in \mathcal{A}$, although we cannot prove it in general case. The problem is that, in contrast to the K -invariant states, the Whittaker vectors do not belong to the L^2 -space. But the case of the $SL(N)$ group (see (I.3.53)) shows that the poles of $h_G^{i\nu-\rho}(yS)$ are canceled by the zeroes of $\exp i\mu_L(n'(yS))$. Thus, the GLWF is also a bounded holomorphic in ϕ and ν solution to the same equation. There is only one solution of such a type. Therefore, up to the normalization, the functions coincide.

Taking the limit $r = r(t) = \exp tH$, $H \in \Lambda^+$, $t \rightarrow +\infty$, we find for $-\nu \in D$ (1.20) the new representation of the standard Harish-Chandra function

$$c(-\nu) \sim \int_N dy h_G^{i\nu-\rho}(yS) \exp i\mu_L(n'(yS)). \quad (2.19)$$

Non-trivial fact is that this integral can be factorized. This can be proved much along the line of [40].

3 Classical reductions

Here we reproduce the classical Hamiltonian description of the open Toda model using the symplectic reductions based on the Iwasawa and the Gauss representations. It allows to write two kinds of actions on the "upstairs" space. It is a classical counterpart of the two Whittaker models described above. In principle, using the functional integral technique, one can calculate the Whittaker wave functions. We will proceed in this way for affine groups.

3.1 Iwasawa reduction

Consider the cotangent bundle T^*G for the real split group G . It is defined by the pair (Y, g) , $Y \in \mathcal{G}^*$, $g \in G$. There is the canonical bi-invariant symplectic form on T^*G

$$\omega = \delta Y(\delta g g^{-1}) \quad (3.1)$$

and the set of invariant commuting Hamiltonians

$$\frac{\langle Y^{d_k} \rangle}{d_k}, \quad k = 1 \dots, r, \quad (3.2)$$

where $d_k = 2, \dots$ are invariants of \mathcal{G} and Y^{d_k} are polynomials on $U(\mathcal{G}^*)$. It is the upstairs Hamiltonian system.

Now consider the gauge transform by the group $K \oplus N$ coming from the Iwasawa decomposition

$$g \mapsto kg, \quad Y \mapsto kYk^{-1}, \quad k \in K \quad (3.3)$$

$$g \mapsto gn, \quad Y \mapsto Y, \quad n \in N.$$

It defines two moment maps

$$\mu_k = Pr_{K^*} Y, \quad \mu_n = Pr_{N^*} g^{-1}Yg, \quad (3.4)$$

where Pr means the orthogonal projection with respect to the Killing form. In the Iwasawa representation, g can be transform by (3.3) to the Cartan subgroup A . Let $g = \exp \phi$, $\phi \in \mathcal{A} = h_I$. Assume that

$$\mu_k = Pr_{K^*} Y = 0, \quad \mu_n = Pr_{N^*} g^{-1}Yg = \mu_I, \quad (3.5)$$

where $\mu_I = \sum_{\alpha \in \Pi} \mu_\alpha \mathcal{G}_\alpha$ is the same as in (2.1). The first relation simply means that $Y \in \mathcal{P}$. The second one can be easily solved and we remain with an undetermined Cartan component. Denote it $p \in \mathcal{A}^*$. We come eventually to the expression

$$Y = p + \sum_{\alpha \in \Pi} \mu_\alpha e^{\alpha(\phi)} (\mathcal{G}_\alpha + \mathcal{G}_{-\alpha}). \quad (3.6)$$

The reduced symplectic form acquires the canonical form

$$\omega^{red} = \delta p \delta \phi = \sum_{k=1}^r \delta p_k \delta \phi_k.$$

The reduced phase space $K \backslash T^*G // N$ has the dimension

$$2 \dim G - 2 \dim K - 2 \dim N = 2r.$$

The combination $W(Y; \tau_1, \dots, \tau_r) = \sum_{k=1}^r \frac{\tau_k}{d_k} \langle Y^{d_k} \rangle$ defines the classical hierarchy of the open Toda lattice. In particular,

$$\langle Y^2 \rangle = \frac{1}{2} \sum_{k=1}^r p_k^2 + \sum_{\alpha \in \Pi} \mu_\alpha e^{2\alpha(\phi)}$$

is the conventional Toda Hamiltonian, which after the canonical quantization coincides with (2.2). The classical action in this representation takes the form

$$S^I = \int Y(\partial_t g g^{-1}) - \langle Y^2 \rangle + \langle B_K, Y \rangle + \langle B_N, g^{-1}Yg - \mu_I \rangle, \quad (3.7)$$

where $B_K \in \mathcal{K}$, $B_N \in \mathcal{N}$ are the Lagrange multipliers.

3.2 Gauss reduction

Consider the symplectic reduction on T^*G with ω (3.1) under the action of $\bar{N} \oplus N$

$$g \mapsto vg, \quad Y \mapsto vYv^{-1}, \quad v \in \bar{N}, \quad (3.8)$$

$$g \mapsto gn, \quad Y \mapsto Y, \quad n \in N.$$

As previously, we have two moment maps

$$\mu_v = Pr_{\bar{N}^*} Y, \quad \mu_n = Pr_{N^*} g^{-1}Yg. \quad (3.9)$$

Using the Gauss decomposition, g can be transform by (3.8) to the Cartan subgroup A . Let $g = \exp \phi$, $\phi \in \mathcal{A}$. Assume that

$$\mu_v = Pr_{\bar{N}^*} Y = \mu^L, \quad \mu_n = Pr_{N^*} g^{-1}Yg = \mu^R, \quad (3.10)$$

where $\mu^L = \sum_{\alpha \in \Pi} \mu_\alpha^L \mathcal{G}_\alpha$, $\mu^R = \sum_{\alpha \in \Pi} \mu_\alpha^R \mathcal{G}_{-\alpha}$ is the same as in ii^G). After "diagonalizing" g by the Gauss representation at the point $g = \exp \phi$, $\phi \in \mathcal{A} = h_G$, we solve constraints (3.10)

$$Y = p + \sum_{\alpha \in \Pi} \left(\mu_\alpha^R e^{\alpha(\phi)} \mathcal{G}_\alpha + \mu_\alpha^L \mathcal{G}_{-\alpha} \right). \quad (3.11)$$

This representation of Y differs from (3.6) by a Cartan gauge transform, and, therefore, yields the Hamiltonian which differs from (2.17) by degrees in the

potential. It corresponds to (2.16). These two expressions of Y are just two different forms of the Lax representation for the Toda lattice.

The classical action in this representation takes the form

$$S^I = \int Y(\partial_t g g^{-1}) - \langle Y^2 \rangle + \langle B_L, Y - \mu^L \rangle + \langle B_R, g^{-1} Y g - \mu^R \rangle, \quad (3.12)$$

where $B_L \in \bar{\mathcal{N}}$, $B_R \in \mathcal{N}$ are the Lagrange multipliers.

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Appendix A

Γ -function formulas used are [35, 39, 38]

$$\begin{aligned} \Gamma(z) &= \frac{b^z}{\cos \frac{\pi z}{2}} \int_0^\infty t^{z-1} \cos bt \, dt, \\ B(x, y) &= \int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}} \, dt = 2 \int_0^\infty \frac{t^{2x-1}}{(1+t^2)^{x+y}} \, dt, \\ B(x, y) &= \int_0^1 (t^{x-1} (1-t)^{y-1}) \, dt, \\ \cos \pi \nu &= \frac{\pi}{\Gamma(\frac{1}{2} + \nu) \Gamma(\frac{1}{2} - \nu)}, \\ \Gamma(\frac{1}{2}) &= \sqrt{\pi}, \\ \Gamma(2x) &= \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma(x + \frac{1}{2}). \end{aligned} \quad (A.1)$$

Cylindric function formulas used are

$$\begin{aligned} K_\nu(z) &= \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\frac{1}{2})} (2z)^\nu \int_0^\infty \frac{\cos t}{(t^2 + z^2)^{\nu + \frac{1}{2}}} \, dt, \\ K_\nu(z) &= \frac{1}{2} \left(\frac{z}{2}\right)^\nu \int_0^\infty \frac{e^{-t - \frac{z^2}{4t}}}{t^{\nu+1}} \, dt, \\ I_\nu(z) &\underset{z \rightarrow 0}{\sim} \frac{1}{\Gamma(\nu + 1)} \left(\frac{z}{2}\right)^\nu. \end{aligned} \quad (A.2)$$

Hypergeometric formulas used are

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int dt t^{b-1} (1-t)^{c-b-1} (1-tx)^{-a}, \quad (A.3)$$

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c; z). \quad (A.4)$$

Integral representation for the Meyer function is

$$G_{pq}^{mn} \left(z \left| \begin{array}{c} a_1, \dots, a_p \\ b_1, \dots, b_q \end{array} \right. \right) = \frac{1}{2\pi i} \int \frac{\Gamma(b_1+s) \dots \Gamma(b_m+s) \Gamma(1-a_1-s) \dots \Gamma(1-a_n-s)}{\Gamma(a_{n+1}+s) \dots \Gamma(a_p+s) \Gamma(1-b_{m+1}-s) \dots \Gamma(1-b_q-s)} z^{-s} ds. \quad (\text{A.5})$$

This function can be also expressed through the finite sums of hypergeometric functions [39].

Appendix B

Proof of (2.15)

Let us prove first that $b(s; \nu, \mu)$ (II.2.12) can be represented as

$$b(s; \nu, \mu) = \frac{1}{\prod_{\alpha \in \Delta^+} \Gamma(1/2 - i\nu_\alpha)} \left(\frac{|\mu_\alpha|}{\sqrt{2} \langle \alpha, \alpha \rangle} \right)^{2 \sum_\gamma -i\nu_\gamma} \prod_{\alpha \in \Delta^+} \Gamma(-is\nu_\alpha). \quad (\text{B.1})$$

Since $M(id; \nu, \mu) = 1$, for $s = id$ (B.1) follows from (II.1.22) and (II.2.13). Assume that (B.1) is valid for all s , $l(s) \leq n$ and let us prove it for $s_\beta s$. It follows from (II.2.11) that

$$b(s_\beta s; \nu, \mu) = \frac{\Gamma(1/2 - is_\beta(s\nu)_\beta)}{\Gamma(1/2 - i(s\nu)_\beta)} \left(\frac{|\mu_\beta|}{\sqrt{2} \langle \beta, \beta \rangle} \right)^{-2i\nu_\beta} M(s; \nu, \mu) c(-s_\beta s\nu). \quad (\text{B.2})$$

Now

$$c(s_\beta s\nu)/c(s\nu) = \frac{\Gamma(1/2 + (s\nu)_\beta) \Gamma(s_\beta(s\nu)_\beta)}{\Gamma(1/2 + s_\beta(s\nu)_\beta) \Gamma((s\nu)_\beta)}. \quad (\text{B.3})$$

This relation can be derived from the following statement:

Each simple reflection s_α maps the subset of positive roots $\Delta^+ \setminus \alpha$ into itself

$$\beta - \frac{2 \langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \alpha \in \Delta^+, \text{ if } \alpha \in \Pi, \beta \in \Delta^+, \text{ and } \beta \neq \alpha.$$

Thereby, all the survived gamma functions in the ratio $c(s_\beta s\nu)/c(s\nu)$ are represented in the right hand side of (B.3). Substituting $c(s_\beta s\nu)$ in (B.2), we come to (B.1).

It is clear that the function $f(\nu) = \prod_{\alpha \in \Delta^+} \sin \pi \nu_\alpha$ is antisymmetric with respect to the action of the Weyl group $f(s\nu) = \det s f(\nu)$, $s \in W$. Now from the relation

$$\frac{\sin \pi \nu_\alpha}{\pi} = [\Gamma(1 - \nu_\alpha) \Gamma(\nu_\alpha)]^{-1}$$

we find that

$$\det s \prod_{\alpha \in \Delta^+} \frac{i \sinh \pi \nu_\alpha}{\pi} = \prod_{\alpha \in \Delta^+} [\Gamma(1 - (s\nu)_\alpha) \Gamma((s\nu)_\alpha)]^{-1},$$

and, therefore,

$$\prod_{\alpha \in \Delta^+} \Gamma(s\nu_\alpha) = \frac{\det s}{\prod_{\alpha \in \Delta^+} \Gamma(1 - (s\nu)_\alpha)} \prod_{\alpha \in \Delta^+} \frac{\pi}{i \sinh \pi \nu_\alpha}.$$

Substituting this expression in (B.1), we find that

$$b(s; \nu, \mu) = b'(s; \nu, \mu) / \xi(\nu),$$

where $\xi(\nu)$ is common multiplier (II.2.14). Thereby, (II.2.15) is proved.

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